CONTENTS

CHAPTER – 1

COMPLEX NUMBERS

INTRODUCTION

We have the knowledge of integers, fractions and irrational number (all these constitute real numbers). But if we try to solve the equation $x^2 + 1 = 0$, we observe that these numbers are not adequate. Trying to solve this equation, we arrive at $x^2 = -1$ i.e. $x = \sqrt{-1}$.

Square of a positive real number is positive and that of a negative real is also positive. So there is no real number whose square is negative. So we are to create a new kind of number. We define the square root of a negative number as imaginary number' particularly $\sqrt{-1} = i$, the basic imaginary number.

Then $\sqrt{-4} = 2i$, $\sqrt{-2} = \sqrt{2}\ i$ and so on .

Imaginary numbers :

Taking $i = \sqrt{-1}$, we observe that

$i^2 = -1$

$i^3 = -1.i = -i$

$i^4 = 1$

Since $i^4 = 1$, $i = i^5 = i^9 = i^{13} = ... = i^{4n+1}$, where n is an integer.

$i^2 = i^6 = i^{10} = i^{14} = ... = i^{4n+2}$

$i^3 = i^7 = i^{11} = i^{15} = ... = i^{4n+3}$

$i^4 = i^8 = i^{12} = i^{16} = ... = i^{4n}$.

COMPLEX NUMBERS

The numbers of the form a + ib where a and b are real numbers and $i = \sqrt{-1}$, are known as complex numbers.

In complex number z = a + ib, the real numbers a and b are respectively know as real and imaginary parts of z and we write :

Re(z) = a and Im (z) = b

Thus the set C of all complex numbers is given by $C = \{z : z = a + ib, \text{ where } a, b \in R\}$

Purely real and purely imaginary numbers :

A complex number z is said to be

(i) Purely real, if Im (z) = 0

(ii) Purely imaginary, if Re (z) = 0

Thus, 2, –7, $\sqrt{3}$ etc are all purely real numbers.

While2i, $i\sqrt{3}$, $\frac{-1}{2}$ i etc are purely imaginary.

Conjugate of a complex number :

The conjugate of a complex number 'z', denoted by $\bar{z}$ is the complex number obtained by changing the sign of imaginary part of z.

e.g. $\left(\overline{2+3i}\right) = (2 - 3i)$; $\left(\overline{3+5i}\right) = (3 - 5i)$,

$\overline{6i} = -6i$; $-\overline{2i} = 2i$

Modulus of a complex number : If z = x + iy be a complex number, the modulus of z, written as | z | is a real number $\sqrt{x^2+y^2}$.

For z = 3 + 4i, | z | = $\sqrt{3^2+4^2}$ = 5.

Also $|\bar{z}|=|z|$.

If z = x + iy, $\bar{z}=x-iy$.

$|z| = \sqrt{x^2+y^2}$, $|\bar{z}|=\sqrt{x^2+(-y)^2}=\sqrt{x^2+y^2}$

SUM DIFFERENCE AND PRODUCT OF COMPLEX NUMBERS

For any complex number

z_1 = (a + ib) and z_2 = (c + id)

we define

(i) $z_1 + z_2$ = (a + ib) + (c + id) = [(a + c) + i (b + d)]

(ii) $z_1 - z_2$ = (a + ib) – (c + id) = [(a – c) + i (b – d)]

(iii) z_1z_2 = (a + ib)(c + id) = [(ac – bd) + i (ad + bc)]

CUBE ROOTS OF UNITY

Let $\sqrt[3]{1}$ = x, then

$x^3 = 1$ [on cubing both sides]

$\Rightarrow$ $x^3 - 1 = 0$ $\Rightarrow$ $(x - 1)(x^2 + x + 1) = 0$

$\Rightarrow$ $x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow$ x = 1 or $x = \dfrac{-1\pm\sqrt{1-4}}{2}$

$\Rightarrow$ x = 1 or $x = \dfrac{-1\pm i\sqrt{3}}{2}$

$\therefore$ The cube roots of unity are 1, $\dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$

Clearly one of the cube roots of unity is real and the other two are complex.

***Example – 1 :* Express in the form a + ib**

(i) $\mathbf{\dfrac{3+5i}{2-3i}}$ **(ii)** $\mathbf{\dfrac{(1+i)^2}{3-i}}$

***Solⁿ :* (i)** $\dfrac{3+5i}{2-3i} = \dfrac{(3+5i)(2+3i)}{(2-3i)(2+3i)} = \dfrac{6+10i+9i+15i^2}{4-9i^2} = \dfrac{-9+19i}{13} = \dfrac{-9}{13} + \dfrac{19}{13}i$

(ii) $\dfrac{(1+i)^2}{3-i} = \dfrac{(1+i^2+2i)(3+i)}{(3-i)(3+i)} = \dfrac{6i-2i^2}{9-i^2} = \dfrac{6i+2}{10} = \dfrac{1}{5} + \dfrac{3}{5}i$

***Example – 2 :* Find the value of $\mathbf{i^{17} + i^{20} - i^{13}}$**

Solⁿ : $i^{17} + i^{20} - i^{13} = i^{16}.\, i + i^{20} - i^{12}.\, i = (i^2)^8.\, i + (i^2)^{10} - (i^2)^6.\, i$

$= (-1)^8 i + (-1)^{10} - (-1)^6 i = i + 1 - i = 1$

***Example – 3 :* If 1, ω, ω² are the cube roots of unity prove that**

(a) $\mathbf{(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9}$

Solⁿ: L.H.S. $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$

$= (1-\omega)(1-\omega^2)(1-\omega^3.\,\omega)(1-\omega^3\omega^2)$

$= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$= (1-\omega)^2(1-\omega^2)^2 = [(1-\omega)(1-\omega^2)]^2$

$= [(1 - \omega - \omega^2 + \omega^3]^2 = (2 - \omega - \omega^2)^2$
$= (2 + 1)^2 = 3^2 = 9$

***Example – 4 :* Find square roots of**
(a) 3 + 4i

***Sol^n^ :* (a)** Let $x, y \in R$, $x + iy = \sqrt{3+4i}$
$x^2 - y^2 + i\,2xy = 3 + 4i$
Equating real and imaginary parts
$x^2 - y^2 = 3$ and $2xy = 4$
$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 25$
Hence $x^2 + y^2 = \pm 5$, But since $x^2 + y^2$ is non-negative, we have
$x^2 + y^2 = 5$
$x^2 - y^2 = 3$
$2x^2 = 8$
i.e, $x^2 = 4$, i.e, $x = \pm 2$, $y^2 = 1$ i.e., $y = \pm 1$
Hence square roots of $3 + 4i = \pm(2 + i)$

Assignment

1. If w be the cube roots of unity, then prove that
$(1 - w + w^2)^7 + (1 + w + w^2)^7 = 128$
2. Find square roots of $-5 + 12\sqrt{-1}$

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CHAPTER – 2

DETERMINANT

INTRODUCTION :

The study of determinants was started by Leibnitz in the concluding portion of seventeenth century. This was latter developed by many mathematician like Cramer, Lagrange, Laplace, Cauchy, Jocobi. Now the determinants are used to study some of aspects of matrices.

Determinant : If the linear equations

$a_1x + b_1 = 0$

and $a_2x + b_2 = 0$

have the same solution, then $\frac{b_1}{a_1} = \frac{b_2}{a_2}$

or $a_1b_2 – a_2b_1 = 0$

The expression $(a_1b_2 – a_2b_1)$ is called a **determinant** and is denoted by symbol.

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ or by (a_1b_2) where a_1, a_2, b_1 & b_2 are called the elements of the **determinant**. The elements in the horizontal direction from rows, and those in the vertical direction form **columns**. The determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has two rows and two coloums. So it is called a **determinant of the second order** and it has 2! = 2 terms in its expansion of which one is positive and other is negative. The diagonal term, or the leading term of the determinant is a_1b_2 whose sign is positive.

Again if the linear equations

$a_1x + b_1y + c_1 = 0$ .... (i)

$a_2x + b_2y + c_2 = 0$ .... (ii)

$a_3x + b_3y + c_3 = 0$ .... (iii)

have the same solutions, we have from the last two equations by cross-multiplication.

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{1}{a_2b_3 - a_3b_2}$$

or $x = \frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}, \ y = \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}$

These values of x and y must satisfy the first equation. Hence $a_1(b_2c_3 – b_3c_2) + b_1 (c_2a_3 – c_3a_2) + c_1 (a_2b_3 – a_3b_2)$

or $a_1b_2c_3 – a_1b_3c_2 + a_3 b_1c_2 – a_2b_1c_3 + a_2b_3c_1 – a_3b_2c_1$ is denoted by the symbol

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ or by $(a_1b_2c_3)$ and has three rows, and three columns. So it is called a **determinant of the third order** and it has 3! = 6 terms of which three terms are positive, and three terms are negative.

MINORS

Minors : The determinant obtained by suppressing the row and the column in which a particular element occurs is called the minor of that element.

Therefore, in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ and that of c_3 is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and so on.

The minor of any element in a third order determinant is thus a second order determinant.

The minors of a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 are denoted by A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 respectively.

Hence $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, A_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \ A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

$B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \ B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \ C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

If Δ stands for the value of the determinant, then

$\Delta = a_1A_1 - b_1B_1 + c_1C_1 = a_1A_1 - a_2A_2 + a_3A_3$

Cofactors : The cofactor of any element in a determinant is its coefficient in the expansion of the determinant. It is therefore equal to the corresponding minor with a proper sign.

For calculation of the proper sign to be attached to the minor of the element, one has to consider $(-1)^{i+j}$ and to multiply this sign with the minor of the element a_{ij} where i and j are respecively the row and the column to which the element a_{ij} belongs.

Thus $C_{ij} = (-1)^{i+j} M_{ij}$ Where C_{ij} and M_{ij} are respectively the cofactor and the minor of the element a_{ij}.

The cofactor of any element is generally denoted by the corresponding capital letter.

Thus for the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, cofactor of a_1 is

$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_1 is $B_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$

that of c_1 is $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

(The sign is $(-1)^{1+3} = 1$), and so on.

We see that minors and cofactors are either equal of differ in sign only.

With this notation the determinant may be expanded in the form,

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1A_1 + b_1B_1 + c_1C_1$

Similarly we express $= a_2A_2 + b_2B_2 + c_2C_2$

$=a_3A_3 + b_3B_3 + c_3C_3$

By expanding with respect to the elements of the first column, we can write

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$= a_1A_1 + a_2A_2 + a_3A_3$

Similarly $= b_1B_1 + b_2B_2 + b_3B_3$

$= c_1C_1 + c_2C_2 + c_3C_3$

Thus the determinant can be expressed as the sum of the product of the elements of any row (or column) and the corresponding cofactors of the respective elements of the same row (or column).

PROPERTIES OF DETERMINANT

I. The value of a determinant is unchanged if the rows are written as columns and columns as rows.

If the rows and coloums are interchanged in the determinant of 2nd order $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, the determinant

becoems $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Each of the two $= a_1b_2 - a_2b_1$

$$\therefore \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

In the third order determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

if the rows and column are interchanged, it

becomes $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta'$ (say)

If Δ is expanded by taking the constituents of the first column and Δ' is expanded by taking the constituents of the first row, then

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

and $\Delta' = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$\therefore \Delta = \Delta'$ (since the value of determinant of 2nd orders is unchanged if rows and columns are interchanged).

II. If two adjacent rows and columns of the determinant are interchanged the sign of the determinant is changed but its absolute value remains unaltered.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Δ' has been obtained by interchanging the first and second rows of Δ
Expanding each determinant by the constituents of the first column.

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

and $\Delta' = a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix}$

$$= -a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\left[\text{since}\begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} + b_2c_1 - c_2b_1 \text{ and } \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + b_1c_2 - b_2c_1\right] = -\Delta$$

In this way it can be proved that only the sign changes if any other two adjacent rows or columns are interchanged.

III. If two rows or columns of a determinant are identical, the determinant vanishes.

Let $\Delta_2 = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$

The first two columns in the determinant are identical. If the first and second columns are interchanged, then the resulting determinant becomes $-\Delta_2$ by II. But since these two columns are identical, the determinant remains unaltered by the interchange.

$\therefore\ \Delta_2 = -\Delta_2$ or, $2\Delta_2 = 0$

$\therefore\ \Delta_2 = 0$

IV. If each constitutent in any row or any column is multiplied by the same factor, then the determinant is multiplied by that factor.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The determinant obtained when the constituents of the first row are multiplied by m is

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1A_1 - ma_2A_2 + ma_3A_3$$

$= m\ [a_1A_1 - a_2A_2 + a_3A_3] = m\Delta$

V. If each constituent in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two other determinants in the determinant.

In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let $a_1 = t_1 + m_1 + n_1$, $a_2 = t_2 + m_2 + n_2$, $a_3 = t_3 + m_3 + n_3$

Then the given determinant

$$= \begin{vmatrix} t_1+m_1+n_1 & b_1 & c_1 \\ t_2+m_2+n_2 & b_2 & c_2 \\ t_3+m_3+n_3 & b_3 & c_3 \end{vmatrix}$$

$= (t_1 + m_1 + n_1)\, A_1 - (t_2 + m_2 + n_2)\, A_2 + (t_3 + m_3 + n_3)\, A_3$

$= (t_1A_1 - t_2A_2 + t_3A_3) + (m_1A_1 - m_2A_2 + m_3A_3) + (n_1A_1 - n_2A_2 + n_3A_3)$

$$= \begin{vmatrix} t_1 & b_1 & c_1 \\ t_2 & b_2 & c_2 \\ t_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} n_1 & b_1 & c_1 \\ n_2 & b_2 & c_2 \\ n_3 & b_3 & c_3 \end{vmatrix}$$

It can be similarly proved that

$$= \begin{vmatrix} a_1+p_1 & b_1+q_1 & c_1 \\ a_2+p_2 & b_2+q_2 & c_2 \\ a_3+p_3 & b_3+q_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & q_1 & c_1 \\ a_2 & q_2 & c_2 \\ a_3 & q_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ P_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 & c_1 \\ p_2 & q_2 & c_2 \\ p_3 & q_3 & c_3 \end{vmatrix}$$

VI. If the constituents of any row (or column) be increased or decreased by any equimultiples of the corresponding constituents of one or more of the other rows (or columns) the value of the determinant remains unaltered.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The determinant obtained, when the constituents of first column are increased by *l* times the second column *m* times the corresponding constituents of the third column is

$$\begin{vmatrix} a_1+lb_1+mc_1 & b_1 & c_1 \\ a_2+lb_2+mc_2 & b_2 & c_2 \\ a_3+lb_3+mc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} lb_1 & b_1 & c_1 \\ lb_2 & b_2 & c_2 \\ lb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mc_1 & b_1 & c_1 \\ mc_2 & b_2 & c_2 \\ mc_3 & b_3 & c_3 \end{vmatrix} \text{(by v)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} \text{(by iv)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

Cramer's Rule :

A method is given below for solving three simultaneous linear equations in three unknowns. This method may also be applied to solve ‘n’ equations in ‘n’ unknowns.

Consider the system of equations.

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{array}\right\} \quad ....(1)$$

Where the coefficients are real.

The coefficient of x, y, z as noted in equations (1) may be used to form the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which is called the determinant of the system.

If $\Delta \neq 0$, the solution of (1) is given by $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}, z = \dfrac{\Delta_3}{\Delta}$, where $\Delta_r; r = 1, 2, 3$ is the determinant obtained from Δ by replacing the r^{th} column by d_1, d_2, d_3.

Example –1 : Find the value of $\begin{vmatrix} \mathbf{5} & \mathbf{-2} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} & \mathbf{2} \\ \mathbf{8} & \mathbf{1} & \mathbf{3} \end{vmatrix}$

Solution : The value of the given determinant

$$= 5\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 8 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 8 & 1 \end{vmatrix}$$

$= 5 (0 – 2) – 2 (9 – 16) + 1 (3 – 0)$

$= – 10 + 14 + 3 = 7$

Example – 2. **Prove that** $\begin{vmatrix} \mathbf{a} & \mathbf{a^2} & \mathbf{a^3} \\ \mathbf{b} & \mathbf{b^2} & \mathbf{b^3} \\ \mathbf{c} & \mathbf{c^2} & \mathbf{c^3} \end{vmatrix}$ **= abc (a – b) (b – c) (c – a)**

Solution : L.H.S. $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (Taking a, b, c, from R_1, R_2, R_3)

$= abc \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$, replacing R_1 by $R_1 – R_2$ and R_2 by $R_2 – R_3$)

$= abc\,(a – b)(b – c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$ (Taking (a – b) & (b – c) common from R_1 & R_2 respectively)

$$= abc\,(a-b)\,(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix} = abc\,(a-b)\,(b-c)\,(c-a)$$

Assignment

1. Find minors & cofactors of the determinants $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

2. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

3. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

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CHAPTER – 3

MATRIX

MATRIX AND ITS ORDER

INTRODUCTION :

In modern engineering mathematics matrix theory is used in various areas. It has special relationship with systems of linear equations which occour in many engineering processes.

A matrix is a reactangular array of numbers arranged in rows (horizontal lines) and columns (vertical lines). If there are ‘m’ rows and ‘n’ Column’s in a matrix, it is called an ‘m’ by ‘n’ matrix or a matrix of order m × n. The first letter in mxn denotes the number of rows and the second letter ‘n’ denotes the number of columns. Generally the capital letters of the alphabet are used to denote matrices and the actual matrix is enclosed in parantheses.

$$\text{Hence } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & -- & a_{1n} \\ a_{21} & a_{22} & a_{23} & -- & a_{2n} \\ a_{31} & a_{32} & a_{33} & -- & a_{3n} \\ -- & -- & -- & -- & -- \\ a_{m1} & a_{m2} & a_{m3} & -- & a_{mn} \end{bmatrix}$$

is a matrix of order m × n and ‘a’$_{ij}$ denotes the element in the ith row and jth column. For example a_{23} is the element in the 2nd row and third column. Thus the matrix ‘A’ may be written as (a_{ij}) where i takes values from 1 to m to represent row and j takes values from 1 to n to represent column.

If m = n, the matrix A is called a square matrix of order n × n (or simply n). Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & -- & --a_{nn} \end{bmatrix}$$

is a square matrix of order n. The determinant of order n, $\begin{vmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & -- & --a_{nn} \end{vmatrix}$

which is associated with the matrix ‘A’ is called the determinant of the matrix and is denoted by det A or |A|.

TYPES OF MATRICES WITH EXAMPLES

(a) **Row Matrix :** A matrix of order 1 × n is called a row matrix. For example (1 2), (a b c) are row matrices of order 1 × 2 and 1 × 3 respectively.

(b) **Column Matrix :** A matrix of order m × 1 is called a column matrix. The matrices $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}a\\b\end{bmatrix}$ are column matrices of order 3 × 1 and 2 × 1 respectively.

(c) **Zero matrix :** If all the elements of a matrix are zero it is called the zero matrix, (or null matrix) denoted by (0). The zero matrix may be of any order. Thus (0), (0, 0), $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ are all zero matrices.

(d) **Unit Matrix :** The square matrix whose elements on its main diagonal (left top to right bottom) are 1's and rest of its elements are 0's is called unit matrix. It is denoted by I and it may be of any order. Thus (1) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ are unit matrices of order 1, 2, 3 respectively.

(e) **Singular and non -singular matrices :** A square matrix A is said to be singular if and only if its determinant is zero and is said to be non-singular (or regular) if det A $\neq$ 0.

For example $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$ is a non singular matrix.

For $\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix} = 4 - 6 = -2 \neq 0$ and $\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}$ is a singular matrix

i.e. $\begin{vmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{vmatrix} = 0$

Adjoint of a Matrix :

The adjoint of a matrix A is the transpose of the matrix obtained replacing each element a_{ij} in A by its cofactor A_{ij}. The adjoint of A is written as adj A. Thus if

$$A = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix} \quad \text{then adj } A = \begin{pmatrix}A_{11} & A_{21} & A_{31}\\A_{12} & A_{22} & A_{32}\\A_{13} & A_{23} & A_{33}\end{pmatrix}$$

***Example – 1 :* Find inverse of the following matrices $\begin{bmatrix}\mathbf{2} & \mathbf{-1}\\\mathbf{1} & \mathbf{3}\end{bmatrix}$**

Sol^n : **(i)** Given A = $\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$, |A| = 7

$A^{-1} = \dfrac{\text{adj } A}{|A|}$, $|A| \neq 0$

So it has inverse

Adj (A)

Minor of 2, $M_{11} = 3$, Cofactor of 2, $C_{11} = 3$

Minor of –1, $M_{12} = 1$, Cofactor of –1, $C_{12} = -1$

Minoir of 1, $M_{21} = -1$, Cofactor of 1, $C_{21} = 1$

Minor of 3, $C_{22} = 2$, Cofactor of 3, $C_{22} = 2$

$$\text{adj}(A) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}}{7} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

Assignment

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

   Calculate (i) AB (ii) BA

2. Find the inverse of the following : $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

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CHAPTER – 4

PARTIAL FRACTIONS

ALGEBRAIC FRACTIONS, PARTIAL FRACTIONS FROM A PROPER FRACTION

Polynomial :

An expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$, where n is a positive integer and $a_0, a_1, a_2 .... a_n$ are real numbers and $a_0 \neq 0$ is called a polynomial of n^{th} degree.

Rational Fraction :

The quotient of two polynomials f(x) and g(x) where $g(x) \neq 0$ is called a rational fraction.

In this section we shall be taking functions which are quotients of two polynomial functions. Such functions are called rational functions.

The functions given by algebric expression such as

$\frac{4x^3-5}{(x-2)^3.(x+2)}$, $\frac{x^2}{(x+1)(x+2)}$, $\frac{2x-5}{x^3-x^2+x-1}$ and $\frac{x^3+x+1}{x^2-1}$ etc are called rational functions. Here both the numerator and denominator are polynomial functions. There are three types of partial fractions.

1. Proper fraction.
2. Improper fraction
3. Mixed fraction.

1. Proper Fraction : If the degree of the numerator is less than the degree of the denominator, the fraction is called proper fraction for e.g. $\frac{1}{(x+1)(x+2)}$, $\frac{2x}{x^2+3x+2}$ and $\frac{x^2}{(x-1)(x-2)(x-3)}$ etc.

$N^0 < D^0$

RESOLVING A RATIONAL FUNCTION INTO PARTIAL FRACTIONS

Case – 1 : When the denominator contains non-repeated linear factors,

for each linear non-repeated factor px + q.

there is partial fraction of the form $\frac{A}{px+q}$.

If $\frac{P(x)}{Q(x)} = \frac{P(x)}{(p_1x+q_1)(p_2x+q_2)(p_3x+q_3).......(p_nx+q_n)}$ is a proper fraction,

then $\frac{P(x)}{Q(x)} = \frac{A_1}{p_1x+q_1} + \frac{A_2}{p_2x+q_2} + \frac{A_3}{p_3x+q_3}$ ..... + $\frac{A_n}{p_nx+q_n}$, where $A_1, A_2, A_3 ...... A_n$ are constants.

Example – 1: **Split $\frac{x}{(x+1)(x+2)}$ into partial fractions**

Sol^n^ : Let $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$= \dfrac{A(x+2)+B(x+1)}{(x+1)(x+2)}$

$\Rightarrow$ $x = A(x + 2) + B(x + 1)$ .... (i)

Putting $x + 2 = 0$ i.e. $x = –2$

$–2 = A \cdot 0 + B(–2 + 1)$

$\Rightarrow$ $–2 = –B$

$\Rightarrow$ $B = 2$

Again put $x + 1 = 0$

$\Rightarrow$ $x = –1$

$\Rightarrow$ $–1 = A(–1 + 2) + B \cdot 0$

$\Rightarrow$ $–1 = A$ $\Rightarrow$ $A = –1$

Putting the values of A & B we get required partial fraction

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

Case – 2: When the denominator contains repeated linear fractors, for a repeated factor like $(px + q)^r$ of the denominator there exists the sum of r partial fractions of the form.

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \frac{A_3}{(px+q)^3} + ... + \frac{A_r}{(px+q)^r}$$

Example – 2 : **Resolve into partial fractions, the function** $\mathbf{\dfrac{1}{(x-1)(x+1)^2}}$

Sol^n^ : Let $\dfrac{1}{(x-1)(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$= \dfrac{A(x+1)^2 + B(x+1)(x-1) + C(x-1)}{(x+1)^2(x-1)}$

$\Rightarrow$ $1 = A(x + 1)^2 + B(x + 1)(x – 1) + C(x – 1)$

Putting $x – 1 = 0$ i.e. $x = 1$,

$1 = A(1 + 1)^2$ $\Rightarrow$ $4A = 1$

$\Rightarrow$ $A = \dfrac{1}{4}$

Putting $x + 1 = 0$ $\Rightarrow$ $x = –1$

$1 = C(–1 – 1)$ $\Rightarrow$ $–2C = 1 \Rightarrow$ $C = -\dfrac{1}{2}$

Equating co-efficients of highest powers of x (i.e. x^2) on both sides in equation

$1 = A(x^2 + 2x + 1) + B (x^2 – 1) + C (x – 1)$

we get $0 = A + B$

i.e $A = –B$ i.e. $B = -\dfrac{1}{4}$

Hence required partial fraction is given by

$$\therefore \frac{1}{(x-1)(x+1)^2}$$

$$= \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2}\frac{1}{(x+1)^2}$$

Case – 3 : When the denominator contains non-repeated quadratic factors which cannot be factorised, For each quadratic non-repeated factor $ax^2 + bx + c$ of the denominator, there exists a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

For example, $\frac{1}{(x^2+\alpha)(x^2+\beta)} = \frac{Ax+B}{x^2+\alpha} + \frac{Cx+D}{x^2+\beta}$

and, $\frac{1}{(x^2+a_1)(x^2+a_2).....(x^2+a_n)}$

$$= \frac{A_1x+B_1}{x^2+a_1} + \frac{A_2x+B_2}{x^2+a_2} + ... + \frac{A_nx+B_n}{x^2+a_n}$$

Example – 3 : **Resolve into partial fractions** $\mathbf{\frac{x}{(x-1)(x^2+1)}}$

Sol^n : Let $\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$$= \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$\Rightarrow$ $x = A(x^2 + 1) + (Bx + C)(x - 1)$ .... (a)

Putting $x - 1 = 0$ i.e. $x = 1$ in (a)

$1 = A(1^2 + 1) + (Bx + C) . 0$

$\Rightarrow$ $2A = 1$ $\Rightarrow$ $A = \frac{1}{2}$

Equating coefficients of highest powers of x on both side in (a)

$x = Ax^2 + A + Bx^2 + Cx - Bx - C$

$0 = A + B$; Equating the coefficients of x^2,

$1 = C - B$; Equating the coefficients of x.

i.e, $A = -B$ i.e. $B = -\frac{1}{2}$

$C - B = 1$

$C + \frac{1}{2} = 1$ $\Rightarrow$ $C = 1 - \frac{1}{2} = \frac{1}{2}$

So its required partial fraction is given by

$$\frac{x}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{(x-1)}{2(x^2+1)}$$

Case – 4 : When the denominator contains repeated quadratic factors,

For each quadratic repeated factor $(ax^2 + bx + c)^r$ of the denominator, there corresponds the sum of r partial fractions of the form.

$$= \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

For example, $\dfrac{1}{(x^2+\alpha)(x^2+\beta)^2} = \dfrac{Ax+B}{(x^2+\alpha)} + \dfrac{Cx+D}{(x^2+\beta)} + \dfrac{Ex+F}{(x^2+\beta)^2}$

$$\frac{1}{(x^2+\alpha)^2(x-\beta)} = \frac{Ax+B}{x^2+\alpha} + \frac{Cx+D}{(x^2+\alpha)^2} + \frac{E}{x-\beta}$$

Assignment

Resolving into partial fractions

1. $\dfrac{84 + 61x - x^2}{(3x+1)(16-x^2)}$

2. $\dfrac{x}{(1+x)(1+x^2)}$

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CHAPTER – 5

BINOMIAL THEOREM

FACTORIAL NOTATION

Let n be a positive integer. Then the product of the numbers $1 \cdot 2 \cdot 3$ .... $(n-1)n$ is called factorial n, and is denoted by n ! or n !.

Thus $n! = 1 \cdot 2 \cdot 3$ .... $(n-1)n$

Ex : $1! = 1$

$2! = 1.2 = 2$

$3! = 1.\ 2.\ 3 = 6$

$4! = 1.\ 2.\ 3.\ 4 = 24$

$5! = 1.\ 2.\ 3.\ 4.\ 5 = 120$

Deduction : $n! = n\,(n-1)\,(n-2)\,(n-3)$ .. 3. 2. 1.

$= n[(n-1)\,(n-2)\,(n-3)$ .... $3.\ 2.\ 1.]$

$= n[(n-1)!]$

Thus $5! = 5 \times (4!)$, $3! = 3 \times (2!)$ & $2! = 2 \times (1!)$

Factorial 'n' is the product of first 'n' natural numbers.

Example – 1 : Prove that :

(i) $n\,(n-1)\,)(n-2)$ .... $(n-r+1) = \dfrac{n!}{(n-r)!}$

Sol^n : (i) $n\,(n-1)\,(n-2)$ .... $(n-r+1)$

$$= \frac{n(n-1)(n-2).......(n-r+1).(n-r)!}{(n-r)!}$$

[Multiplying N^r and D^r by $(n-r)!$] $= \dfrac{n!}{(n-r)!}$

PERMUTATIONS

The different arrangements which can be made out of a given number of things by taking some or all at a time, are called permutations.

Example – 1: All permutations, on arrangements made with the letters a, b, c by taking two at a time are : ab, ba, ac, ca, bc, cb.

Example – 2: All permutations made with the letters a, b, c taking all of at a time are : abc, acb, bac, bca, cab, cba.

Notations : Let r and n be positive integers. Such that $1 \leq r \leq n$

Then the number of different permutations of n dissimilar things, taken r at a time is denoted by P(n, r) or ${}^nP_r$.

$$P(n, r) = n\,(n-1)\,(n-2) ...... (n-r+1) = \frac{n!}{(n-r)!}$$

Note 2 : The number of all permutations of n different things taken all at a time is given by $p(n, n) = n!$

We have $P(n, r) = \frac{n!}{(n-r)!} \Rightarrow \quad P(n, n) = \frac{n!}{0!}$ [Putting r = n]

$\Rightarrow \quad n! = \frac{n!}{0!} \qquad [\because \quad P(n, n) = n!]$

$\Rightarrow \quad 0! = \frac{n!}{n!} = 1$, We are now bound to define 0! = 1,

Each of the different groups of selections which can be formed by taking some or all of numbers of objects, irrespective of their arrangements is called a combination.

Suppose we want to select two out of three persons A, B and C. We may choose AB or BC or AC.

Clearly, AB and BA represent the same selection or group but they give rise to different arrangements. Clearly in a group or selection, the orders in which the objects are arranged is immaterial.

Example – 1 : The different combinations formed of three letters a, b, c taken two at a time are ab, bc, ac.

Example – 2 : The only combination that can be formed of three letters a, b, c taken all at a time is abc.

Example – 3 : Various groups of two out of four persons A, B, C Dare : AB, AC, AD, BC, BD, CD.

BINOMIAL THEOREM

The sum of two quantities a and b (i.e a + b) is called a binomial. Raising it to different powers, we get

$(a + b)^0 = 1, (a + b)^1 = a + b,$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Observe the presence of the co-effieicnts of these expansions in the successive rows of the following triangular arrangement.

Binomial Theorem for positive integral index :

Theorem : If x and y are real numbers, then for all $n \in N$,

$(x + y)^n = C(n, 0)\, x^n + C(n, 1)\, x^{n-1}y + C(n, 2)\, x^{n-2}\, y^2 + \ldots\ldots + C(n, r)\, x^{n-r}\, y^r + \ldots\ldots + C(n, n)\, y^n$

i.e, $(x + y)^n = \sum_{r=0}^{n} C(n,r)\, x^{n-r}\, y^r$

Deduction from Binomial Theorem :

(i) Replacing y by –y, we get :

$(x - y)^n = C(n, 0)\, x^n - C(n, 1)\, x^{n-1}\, y + C(n, 2)\, x^{n-2}\, y^2 + \ldots\ldots + (-1)^n\, C(n, n)\, y^n$

i.e, $(x - y)^n = \sum_{r=0}^{n} (-1)^r . C(n,r)\, x^{n-r}\, y^r$

SOME OBSERVATIONS IN A BINOMIAL EXPANSION

(i) The expansion of $(x + a)^n$ contains (n + 1) terms

(ii) Since C(n, r) = C(n, n – r), It follows that C(n, 0) = C(n, n), C(n, 1) = C(n, n – 1) and so on.

So the coefficient of the terms equidistant form the beginning and the end in a binomial expansion, are equal.

(iii) Middle Terms in a Binomial Expansion :

Since the expansion of $(x + a)^n$ contains (n + 1) terms, so

(a) $\left(\frac{1}{2}n+1\right)^{th}$ terms is the middle term, when n is even.

(b) $\frac{1}{2}$ $(n + 1)^{th}$ term and $\left[\frac{1}{2}(n+1)+1\right]^{th}$ terms are the two middle terms when n is odd.

General term in a binomial expansion :

In a binomial expansion, the (r + 1)th term, i.e., t_{r+1} is taken as the general term.

(i) In the expansion of $(x + y)^n$, we have $t_{r+1} = C(n, r)x^{n-r} y^r$;

(ii) In the expansion of $(x - y)^n$, we have $t_{r+1} = (-1)^r C(n, r)\, x^{n-r} y^r$;

(iii) In the expansion of $(1 + x)^n$, we have $t_{r+1} = C(n, r)\, x^r$,

(iv) In the expansion of $(1 - x)^n$, we have $t_{r+1} = (-1)^r C(n, r)\, x^r$.

Example –1 : **Find the middle terms in the following :**

$$\left(\mathbf{2x^2 - \frac{1}{x}}\right)^7$$

Soln : The number of terms in the expansion is 8. Hence there are two middle terms i.e. 4th and 5th terms.

4th term = $t_4 = t_{3+1} = (-1)^3 C(7, 3)\, (2x^2)^4.\left(\frac{1}{x}\right)^3$

$= -35 \times 16 \times x^8 \times x^{-3} = -560\, x^5$

5th term = $t_5 = t_{4+1} = (-1)^4 C(7, 4)\, (2x^2)^3.\left(\frac{1}{x}\right)^4$

$= 35 \times 8 \times x^6 \times x^{-4} = 280\, x^2$

Assignment

1.Find the coefficients of x^5 in the expansion of $\left(x - \frac{1}{x}\right)^{11}$

2. Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{x^2}\right)^{12}$

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CO-ORDINATE GEOMETRY

CHAPTER – 6

STRAIGHT LINE

CO-ORDINATE SYSTEM

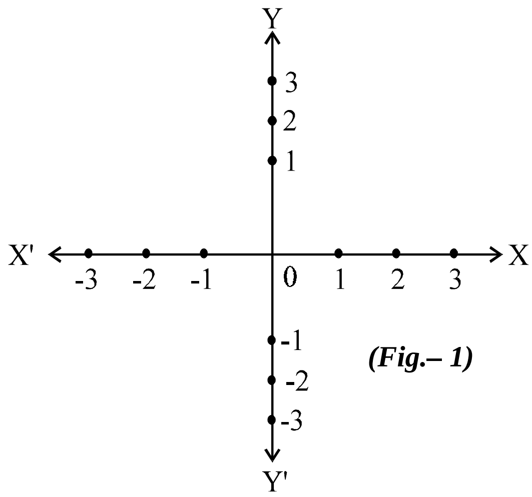
We represent each point in a plane by means of an ordered pair of real numbers, called co-ordinates. The branch of mathematics in which geometrical problems are solved through algebra by using the co-ordinate system, is known as co-ordinate geometry or analytical geometry.

Rectangular co-ordinate Axes

Let X'OX and YOY' be two mutually perpendicular lines (called co-ordinate axes), intersecting at the point O. ***(Fig.1)***.We call the point O, the origin, the horizontal line X'OX, the x-axis and the vertical line YOY', the y-axis.

We fix up a convenient unit of length and starting from the origin as zero, mark. distances on x-axis as well as y-axis.

The distance measured along OX and OY are taken as positive while those along OX' and OY' are considered negative.



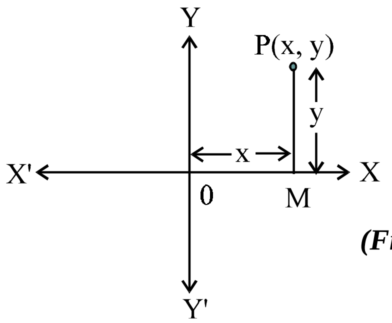
(Fig.– 1)

Cartesian co-ordinates of a point

Let X'OX and YOY' be the co-ordinate axes and let P be a point in the Euclidean plane ***(Fig.2)***. From P draw PM $\perp$ X'OX.

Let OM = x and PM = y, Then the ordered pair (x, y) represents the cartesian co-ordinates of P and we denote the point by P(x, y). The number x is called the x-co-ordinate or abscissa of the point P, while y is known as its y-co-ordinate or ordinate.

Thus, for a given point the abscissa and the ordinate are the distances of the given point from y- axis and x-axis respectively.



(Fig.– 2)

Quadrants

The co-ordinate axes X'OX and Y'OY divide the plane in to four regions, called quadrants.

The regions XOY, YOX', X'OY' and Y'OX are known as the first, the second, the third and the fourth quadrant respectively. ***(Fig.3)***

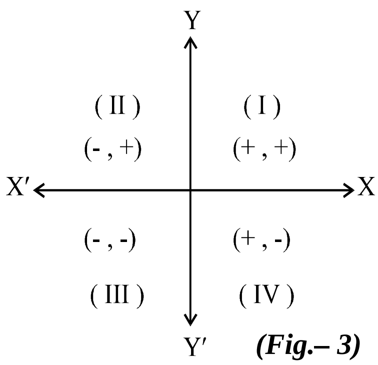
In accordance with the convention of signs defined above for a point (x, y) in different quadrants we have

1st quadrant : $x > 0$ and $y > 0$

2nd quadrant : $x < 0$ and $y > 0$

3rd quadrant : $x < 0$ and $y < 0$

4th quadrant : $x > 0$ and $y < 0$



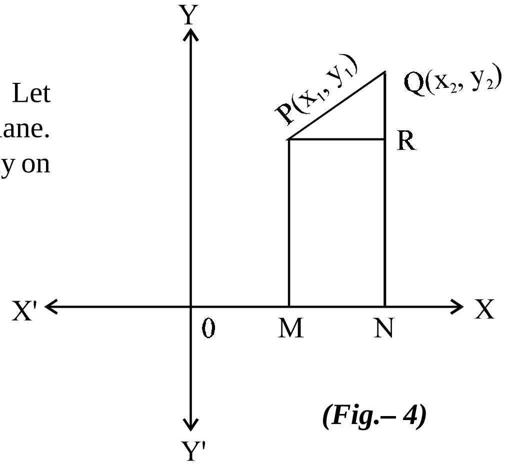
(Fig.– 3)

DISTANCE BETWEEN TWO GIVEN POINTS

The distance between any two points in the plane is the length of the line segment joining them.

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$$



(Fig.– 4)

Proof : Let X'OX and YOY' be the co-ordinate axes ***(Fig.4)***. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points in the plane. From P and Q draw perpendicular PM and QN respectively on the x-axis. Also draw PR $\perp$ QN.

Then, $OM = x_1$, $ON = x_2$

$PM = y_1$ & $QN = y_2$

$\therefore$ $PR = MN = ON - OM = x_2 - x_1$

and $QR = QN - RN = QN - PM = y_2 - y_1$

Now from right angled triangle PQR,

we have $PQ^2 = PR^2 + QR^2$ [by Pythagoras theorem]

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

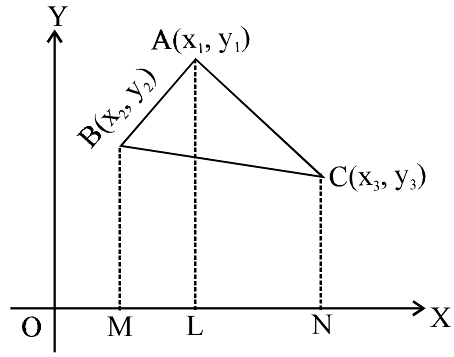
$\therefore$ $|PQ| = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$

Cor : The distance of a point P(x, y) from the origin O (0, 0) is

$$= \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Area of a triangle :

Let ABC be a given triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. From the vertices A, B and C draw perpendiculars AL, BM and CN respectively on x-axis. ***(Fig.5)***.



(Fig.– 5)

Then, $ML = x_1 - x_2$; $LN = x_3 - x_1$ and $MN = x_3 - x_2$

$\therefore$ Area of Δ ABC

= area of trapezium ALMB + area of trapezium ALNC – area of trapezium BMNC

$$= \frac{1}{2}(AL + BM) \cdot ML + \frac{1}{2}(AL + CN) \cdot LN - \frac{1}{2}(MB + CN) \cdot MN$$

$$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2 + x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 - x_3y_2 - x_3y_3 + x_2y_2 + x_2y_3]$$

$$= \frac{1}{2}[x_1y_2 - x_2y_1 + x_3y_1 - x_1y_3 - x_3y_2 + x_2y_3]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

In determinant form, we may write

$$\text{Area of } \Delta\,ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition for collinearity of Three points :

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are colliner, i.e. lie on the same straight line, if the area of $\Delta\,ABC$ is zero. So the required condition for A, B, C to be collinear is that

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$$

$$\Rightarrow \quad x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

Formula for Internal Divisions :

The co-ordinates of a point P which divides the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio m : n are given by

$$\bar{x} = \frac{mx_2 + nx_1}{m+n}, \quad \bar{y} = \frac{my_2 + ny_1}{m+n}$$

Example – 1 : **In what ratio does the point (3, –2) divide the line segment joining the points (1, 4) and (–3, 16) :**

Solution : Let the point C (3, –2) divide the segment joining A(1, 4) and B (–3, 16) in the ratio K: 1

The co-ordinates of 'C' are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1}\right)$

But we are given that the point C is (3, –2)

$\therefore$ We have $\frac{-3k+1}{k+1} = 3$

or $-3k + 1 = 3k + 3$

or $-6k = 2$

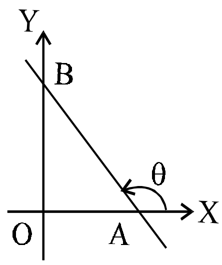
$\therefore$ $k = -\frac{1}{3}$

$\therefore$ C divides AB in the ratio 1 : 3 externally.

SLOPE OF A LINE

Angle of Inclination : The angle of inclination or simply the inclination of a line is the angle θ made by the line with positive direction of x-axis, measured from it in anticlock wise direction ***(Fig. 6)***.

Slope or gradient of a line : If θ is the inclination of a line, then the value of $\tan\theta$ is called the slope of the line and is denoted by m.



(Fig.– 6)

CONDITIONS OF PARALLELISM AND PERPENDICULARITY

1. Two lines are parallel if and only if their slopes are equal.

2. Two lines with slope m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$

3. The slope of a line passing through two given points (x_1, y_1) and (x_2, y_2) is given by $m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

4. The equation of a line with slope m and making an intercept 'c' on y-axis is given by y = mx + c.

Proof : Let AB be the given line with inclination θ so that $\tan\theta = m$. Let it intersect the y-axis at C so that OC = c. ***(Fig.7)***

Let it intersect the x-axis at A.

Let P(x, y) be any point on the line.

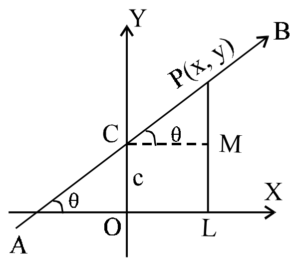
Draw PL perpendicular to x-axis and CM $\perp$ PL

Clearly, $\angle MCP = \angle OAC = \theta$

CM = OL = x ;

and PM = PL – ML = PL – OC = y – c

Now, from rt. angled Δ PMC



(Fig.– 7)

We get $\tan\theta = \dfrac{PM}{CM}$ or $m = \dfrac{y-c}{x}$

or $y = mx + c$, which is required equation of the line.

5. The equation of a line with slope m and passing through a point (x_1, y_1) is given by $(y - y_1) = m(x - x_1)$

6. The equation of a line through two given points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

7. The equation of a straight line which makes intercepts of length 'a' and 'b' on x-axis and y-axis respectively, is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Proof : Let AB be a given line meeting the x-axis and y-axis at A and B respectively ***(Fig.8)***.

Let OA = a and OB = b

Then the co-ordinates of A, B are A(a, 0) and B(0, b)

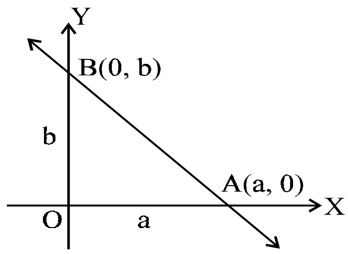
$\therefore$ The equation of the line joining A & B is

$$(y - 0) = \frac{b-0}{0-a}(x - a)$$

$$\Rightarrow \quad y = \frac{-b}{a}(x - a)$$

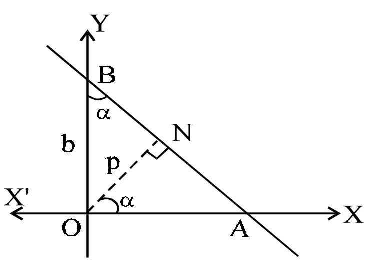
$$\Rightarrow \quad \frac{y}{b} = \frac{-x}{a} + 1$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$



(Fig.– 8)

8. Let P be the length of perpendicular from the origin to a given line and α be the angle made by this perpendicular with the positive direction of x-axis. Then the equation of the line is given by $x \cos\alpha + y \sin\alpha = P$



(Fig.– 9)

Conditions for two lines to be coincident, parallel, perpendicular or Intersect :

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) conicident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$;

(ii) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Perpendicular, if $a_1a_2 + b_1b_2 = 0$;

(iv) Intersecting, if they are neither coincident nor parallel.

Example – 1 : **Find the equation of the line which passes through the point (3, 4) and the sum of its intercept on the axes is 14.**

Sol^n : Let the intercept made by the line on x-axis be 'a' and 'y'- axis be 'b'

i.e. $a + b = 14$ i.e, $b = 14 - a$

$\therefore$ Equation of the line is given by

$$\frac{x}{a} + \frac{y}{14-a} = 1 \text{ .... (i)}$$

As the point (3, 4) lies on it, we have

$$\frac{3}{a} + \frac{4}{14-a} = 1$$

or $3(14 - a) + 4a = 14a - a^2$

or $42 - 3a + 4a = 14a - a^2$

or $a^2 - 13a + 42 = 0$

or $(a - 7)(a - 6) = 0$

or $a = 7$ or $a = 6$

Putting these values of a in (i)

$$\frac{x}{7} + \frac{y}{7} = 1 \quad \text{or} \quad x + y = 7$$

and $\frac{x}{6} + \frac{y}{8} = 1$ or $4x + 3y = 24$

Example – 2 : **Find the equation of the line passing through (–4, 2) and parallel to the line $4x - 3y = 0$**

Sol^n : Any line passing thorugh (–4, 2) whose equation is given by

$(y - 2) = m(x + 4)$ .... (i)

and parallel to the given line $4x - 3y = 0$

whose slope is $y = \frac{4}{3}x$

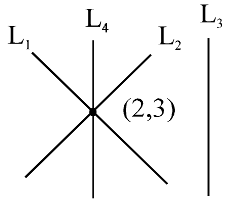
Here 'm' = $\frac{4}{3}$

It's equation is

$(y - 2) = \frac{4}{3}(x + 4)$

$3y - 6 = 4x + 16$

or $4x - 3y + 22 = 0$



(Fig.– 10)

Example – 3 : **Find the equation of the line passing through the intersection of 2x – y – 1 = 0 and 3x – 4y + 6 = 0 and parallel to the line x + y – 2 = 0**

Sol**^{n} **: Point of intersection of 2x – y – 1 = 0 and 3x – 4y + 6 = 0

$$\left(\frac{-1\times 6-(-4)(-1)}{2(-4)-3(-1)}, \frac{(-1)\times 3-6(2)}{2(-4)-3(-1)}\right)$$

$$= \left(\frac{-6-4}{-8+3}, \frac{-3-12}{-8+3}\right) = \left(\frac{-10}{-5}, \frac{-15}{-5}\right) = (2, 3)$$

Any line parallel to the line x + y – 2 is given by x + y + k = 0.... (i)

Since the line passes through (2, 3) hence it satisfies the equation (i)

So, 2 + 3 + k = 0

$\Rightarrow$ k = –5

Now putting the value of k in equation (i), we get x + y – 5 = 0

$\therefore$ Equation of the line is x + y – 5 = 0

Assignment

1. Find the equation of a line parallel to 2x + 4y – 9 = 0 and passing through the point (–2, 4)
2. Find the co-ordinates of the foot of the perpendicular from the point (2, 3) on the line 3x – 4y + 7 = 0
3. Find the equation of the line through the point of intersection of 3x + 4y – 7 = 0 and x – y + 2 = 0 and which is parallel to the line 5x – y + 11 = 0

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CHAPTER – 7

CIRCLE

A circle is the locus of a point which moves in a plane in such a way that it's distance from a fixed point is always constant.

The fixed point is called the centre of the circle and the constant distance is called its radius.

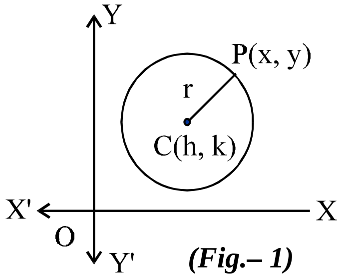
Equation of a circle (Standard form)

Let C(h, k) be the centre of a circle with radius 'r' and let P (x, y) be any point on the circle ***(Fig.1)***.

Then CP = r $\Rightarrow$ $CP^2 = r^2$

$\Rightarrow (x - h)^2 + (y - k)^2 = r^2$

Which is required equation of the circle.



(Fig.– 1)

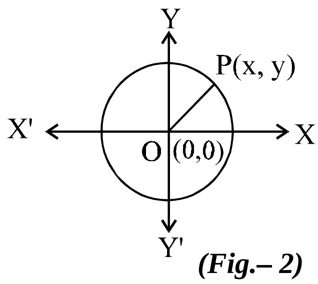
Cor. The equation of a circle with the centre at the origin and radius r, is $x^2 + y^2 = r^2$ ***(Fig.2)***.

Proof : Let O (0, 0) be the centre and r be the radius of a circle and let P (x, y) be any point on the circle.

Then OP = r $\Rightarrow$ $OP^2 = r^2$

$\Rightarrow (x - 0)^2 + (y - 0)^2 = r^2$

$\Rightarrow x^2 + y^2 = r^2$



(Fig.– 2)

***Example – 1.* Find the equation of a circle with centre (–3, 2) and radius 7.**

***Sol^n* :** The required equation of the circle is

$[x - (-3)]^2 + (y - 2)^2 = 7^2$

or $(x + 3)^2 + (y - 2)^2 = 49$

or $x^2 + y^2 + 6x - 4y - 36 = 0$

***Example – 2.* Find the equation of a circle whose centre is (2, –1) and which passes through (3, 6)**

***Sol^n* :** Since the point P (3, 6) lies on the circle, its distance from the centre C (2, –1) is therefore equal to the radius of the circle.

$\therefore$ Radius = CP = $\sqrt{(3-2)^2 + (6+1)^2} = \sqrt{50}$

So, the required equation of the circle is

$(x - 2)^2 + (y + 1)^2 = 50$

or $x^2 + y^2 - 4x + 2y - 45 = 0$

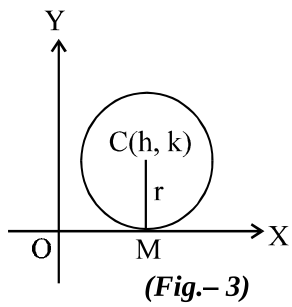
***Example – 3.* Find the equation of a circle with centre (h, k) and touching the x-axis *(Fig.3)*.**

***Sol^n* :** Clearly, the radius of the circle = CM = r = k

So, the required equation

$(x - h)^2 + (y - k)^2 = k^2$

or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$



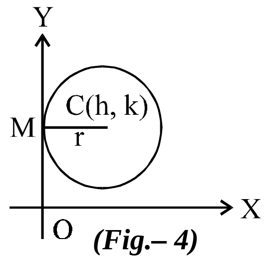
(Fig.– 3)

***Example – 4*. Find the equation of a circle with centre (h,k) and touching y-axis*(Fig.4)*.**

***Sol^n* :** Clearly, the radius of the circle = CM = r = h

So, the required equation is $(x - h)^2 + (y - k)^2 = h^2$

or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$



(Fig.– 4)

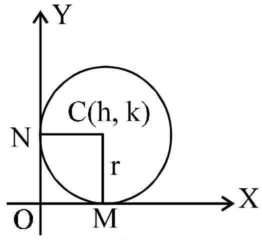
***Example – 5*. Find the equation of a circle with centre (h,k) and touching both the axes *(Fig.5)*.**

***Sol^n* :** Clearly, radius, CM = CN = r

i.e. h = k = r (say)

$\therefore$ the equation of the circle is $(x - r)^2 + (y - r)^2 = r^2$

or $x^2 + y^2 - 2r(x + y) + r^2 = 0$



(Fig.– 5)

GENERAL EQUATION OF A CIRCLE

Theorem : The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$
And, every such equation represents a circle.

Proof : The standard equation of a circle with centre (h, k) and radius r is given by

$(x - h)^2 + (y - k)^2 = r^2$

Or $x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

This is of the form

$x^2 + y^2 + 2gx + 2fy + c = 0$

Where $h = -g, k = -f$ and $c = (h^2 + k^2 - r^2)$

Conversely, let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the given condition.

Then, $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = (g^2 + f^2 - c)$

$\Rightarrow (x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

$\Rightarrow [x - (-g)]^2 + [y - (-f)]^2 = \left[\sqrt{g^2 + f^2 - c}\right]^2$

$\Rightarrow (x - h)^2 + (y - k)^2 = r^2$

Where $h = -g$, $k = -f$ and $r = \sqrt{g^2 + f^2 - c}$

This shows that the given equation represents a circle with centre (–g, –f) and radius.

$= \sqrt{g^2 + f^2 - c}$, provided $g^2 + f^2 > c$.

EQUATION OF A CIRCLE WITH GIVEN END POINTS OF A DIAMETER

Theorem : The equation of a circle described on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_1)$ as a diameter, is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

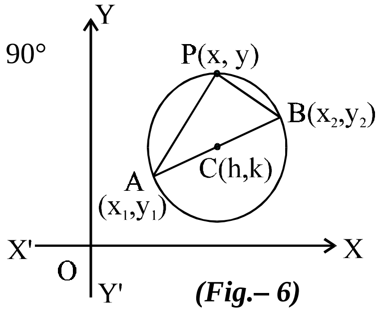
Proof : Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end point of a diameter of the given circle and let P (x, y) be any point on the circle ***(Fig.6)***.

Since the angle in a semi-circle is a right angle, we have $\angle APB = 90°$

Now slope of AP = $\left[\dfrac{y - y_1}{x - x_1}\right]$

And, slope of BP = $\left(\dfrac{y - y_2}{x - x_2}\right)$

Since AP $\perp$ BP, we have



(Fig.– 6)

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_1}\right)=-1$$

Or $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

***Example – 1.* Find the equation of a circle whose end points of diameter are (3, 4) and (–3, –4)**

Sol^n. : The required equation of the circle is $(x - 3)(x + 3) + (y - 4)(y + 4) = 0$

i.e. $x^2 - 9 + y^2 - 16 = 0$

or $x^2 + y^2 = 25$

***Example – 2.* Find the centre and radius of the circle.**

$$\mathbf{x^2 + y^2 - 6x + 4y - 36 = 0}$$

***Sol^n*. :** Comparing the equation with

$x^2 + y^2 + 2gx + 2fy + c = 0$

We get $2g = -6$, $2f = 4$ and $c = -36$

or $g = -3$, $f = 2$ and $c = -36$

$\therefore$ Centre of the circle is $(-g, -f)$, i.e. $(3, -2)$

And radius of the circle.

$= \sqrt{g^2+f^2-c}=\sqrt{9+4+36}=7$

Assignment

1. Find the centre and radius of each of the following circles
   $x^2 + y^2 + x - y - 4 = 0$
2. Find the equation of the circle whose centre is (–2, 3) and passing through origin
4. Find the equation of the circle having centre at (1, 4) and passing through (–2, 1).
4. Find the equation of the circle passing through the points (1, 3) (2, –1) and (–1, 1).

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TRIGONOMETRY

CHAPTER – 8

COMPOUND ANGLES

INTRODUCTION :

The word Trigonometry is derived from Greek words ‘‘Trigonos’’ and metrons means measurement of angles in a triangle. This subject was originally devecpaed to solve geometric problems involving trigangles. The Hindu mathematicians Aryabhatta, Varahmira, Bramhaguptu and Bhaskar have lot of contaribution to trigonometry . Besides Hindu mathematicians ancient-Greek and Arwric mathematicians also contributed a lot to this subject. Trigonometry is used in many are as such as science of seismology, designing electrical circuits, analysing musical tones and studying the occurance of sun spots.

Trigonometric Functions :

Let θ be the meausre of any angle measured in radians in counter clockmise sense as show in Fig (1).

Let P(x, y) be any point an the terminal side of angle θ. The distance of P from O is OP = r = $\sqrt{x^2+y^2}$. the functions defined by $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$ ...(1) are called sine, cosine and tangent functions respectirely. These are called trigonometric functions. It followrs from (1) that $\sin^2\theta + \cos^2\theta = 1$. Other trigonomatric functions such as cosecant, secant and cotangent functions are defined as $\operatorname{cosec}\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$.

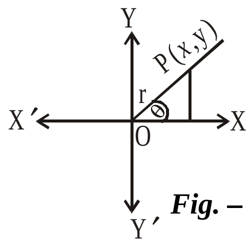
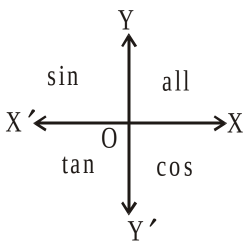


Fig. – 1

SIGN OF T-RATIOS :

The student may remember the signs of t-ratios in different quadrant with the help of the diagram



The sign of paricular t-ratio in any quadrant can be remembered by the word ‘‘all-sin-tan-cos’’ or ‘‘add sugar to coffee’’. What ever is written in a particular quadrant along with its reciprocal is +ve and the rest are negetive.

Table giving the values of trigonometrical Ratios of angles 0º, 30°, 45º, 60º & 90º

θ	0º	30º	45º	60º	90º
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

RELATED ANGLES :

Definitions : Two angles are said to be complementary angles if their sum is 90° and each angle is said to be the complement of the other.

Two angles are said to be supplementary if their sum is 180° and each angles is said to be the supplement of the other.

To Find the T-Ratios of angle (–θ) in terms of θ :

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle θ in the anticlockwise sense which we take as positive sense. ***(Fig. 2)***

Let OP' be the position of the radius vector after tracing (θ) in the clockwise sense, which we take as negative sense. So $\angle P'OX$ will be taken as –θ. Join PP'. Let it meet OX at M.

Now Δ OPM $\equiv$ Δ P'OM, $\angle P'OM = -\theta$

OP' = OP, P'M = – PM

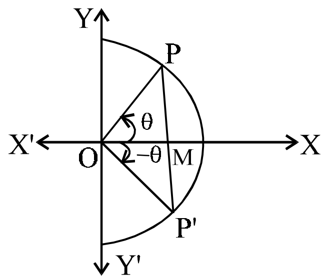


Fig. – 2

Now $\sin(-\theta) = \frac{P'M}{OP'} = \frac{-PM}{OP} = -\sin\theta$

$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta$

$\tan(-\theta) = \frac{P'M}{OM} = \frac{-PM}{OM} = -\tan\theta$

$\operatorname{cosec}(-\theta) = \frac{OP'}{P'M} = \frac{OP}{-PM} = -\operatorname{cosec}\theta$

$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta$

$\cot(-\theta) = \frac{OM}{P'M} = \frac{OM}{-PM} = -\cot\theta$

To find the T-Ratios of angle (90° – θ) in terms of θ.

Let OPM be a right angled triangle with $\angle POM = 90°$, $\angle OMP = \theta$, $\angle OPM = 90° - \theta$. ***(Fig. 3)***

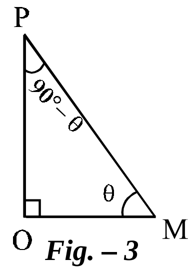


Fig. – 3

$\therefore$ $\sin(90° - \theta) = \frac{OM}{PM} = \cos\theta \Rightarrow \operatorname{cosec}(90° - \theta) = \sec\theta$

$\cos(90° - \theta) = \frac{OP}{PM} = \sin\theta \Rightarrow \sec(90° - \theta) = \operatorname{cosec}\theta$

$$\tan(90° - \theta) = \frac{OM}{OP} = \cot\theta \quad \Rightarrow \quad \cot(90° - \theta) = \tan\theta$$

To find the T-Ratios of angle (90° + θ) in terms of θ.

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$. Draw PM and P'M' perpendiculars to the X-axis***(Fig. 4)***

Now $\Delta POM \cong \Delta P'OM'$

$\therefore$ P'M' = OM and OM' = –PM

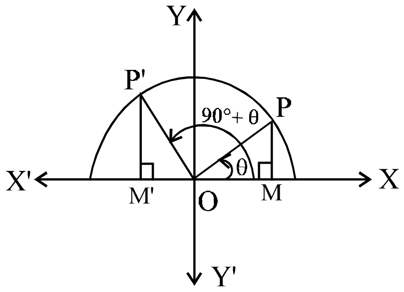


Fig. – 4

Now $\sin(90° + \theta) = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos\theta$

$$\cos(90° + \theta) = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin\theta$$

$$\tan(90° + \theta) = \frac{P'M'}{OM'} = \frac{OM}{-PM} = -\cot\theta$$

Similarly cosec (90° + θ) = sec θ

sec (90° + θ) = – cosec θ

and cot (90° + θ) = – tan θ

To Find the T-Ratios of angle (180° – θ) in terms of θ.

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle XOP = θ To obtain the angle 180° – θ let the radius vector start from OX and after revolving through 180° come to the position OX'. Let it revolve back through an angle θ in the clockwise direction and come to the position OP' so that the angle X'OP' is equal in magnitude but opposite in sign to the angle XOP. The angle XOP' is 180° – θ. ***(Fig.5)***

Draw P'M' and PM perpendicular to X'OX.

Now $\Delta POM \cong \Delta P'OM'$.

$\therefore$ OM' = –OM and P'M' = PM

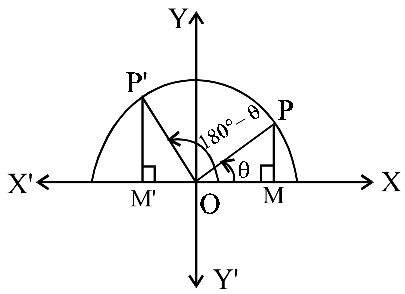


Fig. – 5

Now $\sin(180° - \theta) = \frac{P'M'}{OP'} = \frac{PM}{OP} = \sin\theta$

$$\cos(180° - \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{P'M'}{OM'} = \frac{PM}{-OM} = -\tan\theta$$

Similarly cosec (180° – θ) = cosec θ

sec (180° – θ) = –sec θ

and cot (180° – θ) = – cot θ

To Find the T-Ratios of angle (180° + θ) in terms of θ.

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$.***(Fig. 6)***

Now $\Delta POM \cong \Delta P'OM'$.

$\therefore$ OM' = –OM and P'M' = –PM

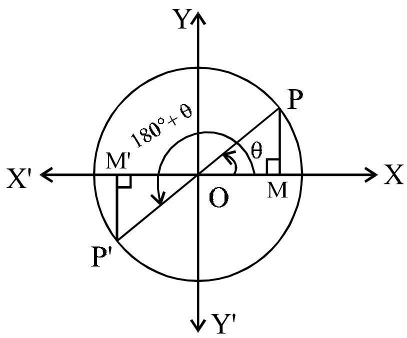


Fig. – 6

Now $\sin(180° + \theta) = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin\theta$

$$\cos(180° + \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$$

$$\tan(180° + \theta) = \frac{P'M'}{OM'} = \frac{-PM}{-OM} = \tan\theta$$

Similarly cosec (180° + θ) = cosec θ

sec (180° + θ) = –sec θ

and cot (180° + θ) = cot θ.

To Find the T-Ratios of angles (270° $\pm$ θ) in terms of θ.

The trigonometrical ratios of 270° – θ and 270° + θ in terms of those of θ, can be deduced from the above articles. For example

sin (270° – θ) = sin [180° + (90° – θ)]

= –sin (90° – θ) = – cos θ

cos (270° – θ) = cos [180° + (90° – θ)]

= – cos (90° – θ) = – sin θ

Similarly sin (270° + θ) = sin [180° + (90° + θ)]

= –sin (90° + θ) = – cos θ

cos (270° + θ) = cos [180° + (90° + θ)]

= – cos (90° + θ) = – (–sin θ) = sin θ

To Find the T-Ratios of angles (360° $\pm$ θ) in terms of θ.

We have seen that if n is any integer, the angle n. 360° $\pm$ θ is represented by the same position of the radius vector as the angle $\pm$ θ . Hence the trigonometrical ratios of 360° $\pm$θ are the same as those of $\pm$θ.

Thus sin (n. 360° + θ) = sin θ

cos (n. 360° + θ) = cos θ

sin (n. 360° – θ) = sin (–θ) = – sin θ

and cos (n. 360° – θ) = cos (–θ) = cos θ.

Examples :

cos (–720° – θ) = cos (–2 × 360° – θ) = cos (–θ) = cos θ

and tan (1440° + θ) = tan (4 × 360° + θ) = tan θ

In general when is any integer, n $\in$ Z

(1) $\sin(n\pi + \theta) = (-1)^n \sin\theta$

(2) $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(3) $\tan(n\pi + \theta) = \tan\theta$ when n is odd integer

(4) $\sin\left(\frac{n\pi}{z} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$

(5) $\cos\left(\frac{n\pi}{z} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$

(6) $\tan\left(\frac{n\pi}{z} + \theta\right) = \cot\theta$

EVEN FUNCTION :

A function $f(x)$ is said to be an even function of x, if $f(x)$ satisfies the relation $f(-x) = f(x)$.

Ex. cosx, secx, and all even powers of x i.e, x^2, x^4, x^6...... are even function.

ODD FUNCTION :

A function $f(x)$ is said to be an odd function of x, if $f(x)$ satisfies the relation $f(-x) = -f(x)$.

Ex. sinx, cosec x, tan x, cot x and all odd powers of x i.e, x^3, x^5, x^7...... are odd function.

Example : ***Find the values of sinθ and tanθ if cosθ = $\frac{-12}{13}$ and θ lies in the third quadrant.***

Solution : We have $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$$

In third quadrant sinθ is negetive, therefore

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(\frac{-12}{13}\right)^2} = \frac{-5}{3}$$

Now $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-5}{13} \times \dfrac{13}{-12} = \dfrac{5}{12}$

Example : Find the values of

(i) tan (–900º) (ii) sin 1230º

Solution : (i) tan (–900º) = –tan 900º = – tan (10 × 90º + 0º) = – tan0º = 0

(ii) sin (1230º) = sin (6 × 180º + 150º) = sin 150º = sin (180º – 30º) = sin 30º = $\frac{1}{2}$

Example : Show that

$$\frac{\cos(90^\circ+\theta).\sec(-\theta).\tan(180^\circ-\theta)}{\sec(360^\circ-\theta).\sin(180^\circ+\theta).\cot(90^\circ-\theta)} = -1 = \frac{-\sin\theta\times\sec\theta\times-\tan\theta}{\sec\theta\times-\sin\theta\times\tan\theta} = -1$$

Solution : $\dfrac{\cos(90^\circ+\theta).\sec(-\theta).\tan(180^\circ-\theta)}{\sec(360^\circ-\theta).\sin(180^\circ+\theta).\cot(90^\circ-\theta)} = \dfrac{-\sin\theta\times\sec\theta\times-\tan\theta}{\sec\theta\times-\sin\theta\times\tan\theta} = -1$

ASSIGNMENT

1. Find the value of cos1º.cos2º..... cos 100º

2. Evaluale : $\tan\frac{\pi}{20}\cdot\tan\frac{3\pi}{20}\cdot\tan\frac{5\pi}{20}\cdot\tan\frac{7\pi}{20}\cdot\tan\frac{9\pi}{20}\cdot$

❒ ☸ ❒

COMPOUND, MULTIPLE AND SUB-MULTIPLE ANGLES

When an angle formed as the algebric sum of two or more angles is called a compound angles. Thus A + B and A + B + c are compound angles.

Addition Formulae

When an angle formed as the algebraical sum of two or more angles, it is called a compound angles. Thus A + B and A + B + C are compound angles.

Addition Formula :

(i) sin (A + B) = sin A . cos B + cos A . sin B

(ii) cos (A + B) = cos A . cos B – sin A . sin B

(iii) tan (A + B) = $\dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$

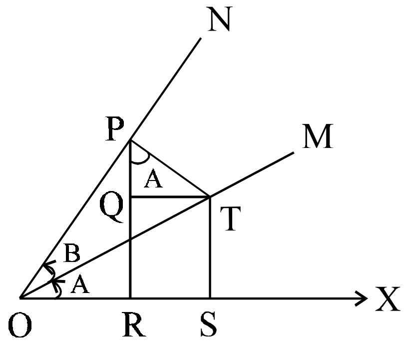


Fig. – 7

Proof : Let the revolving line OM starting from the line OX make an angle XOM = A and then further move to make.

$\angle$ MON = B, so that $\angle$ XON = A + B ***(Fig. 7)***

Let 'P' be any point on the line ON.

Draw $PR \perp OX$, $PT \perp OM$, $TQ \perp PR$ and $TS \perp OX$

Then $\angle QPT = 90° - \angle PTQ = \angle QTO = \angle XOM = A$

$\therefore$ We have from Δ OPR

(i) sin (A + B) = $\dfrac{RP}{OP} = \dfrac{QR + PQ}{OP} = \dfrac{TS + PQ}{OP}$ ($\because$ QR = TS)

$= \dfrac{TS}{OP} + \dfrac{PQ}{OP} = \dfrac{TS}{OT} . \dfrac{OT}{OP} + \dfrac{PQ}{PT} . \dfrac{PT}{OP}$

= sin A . cos B + cos A . sin B

(ii) cos (A + B) = $\dfrac{OR}{OP} = \dfrac{OS - RS}{OP} = \dfrac{OS}{OP} - \dfrac{RS}{OP}$

$= \dfrac{OS}{OP} - \dfrac{QT}{OP}$ [$\because$ RS = QT]

$= \dfrac{OS}{OT} . \dfrac{OT}{OP} - \dfrac{QT}{PT} . \dfrac{PT}{OP}$

= cos A . cos B – sin A . sin B

(iii) tan (A + B) = $\dfrac{\sin(A + B)}{\cos(A + B)}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B)}$

(dividing numerator and denominator by cos A cos B)

$$= \dfrac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$$

(iv) $\cot(A+B) = \dfrac{\cos(A+B)}{\sin(A+B)}$

$= \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

[dividing of numerator and denominator by sin A sin B]

$= \dfrac{\dfrac{\cos A \cos B}{\sin A \sin B} - 1}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}}$

$\cot(A+B) = \dfrac{\cot A . \cot B - 1}{\cot B + \cot A}$

Cor : In the above formulae, replacing A by $\dfrac{\pi}{2}$ and B by x

We have

(i) $\sin\left(\dfrac{\pi}{2} + x\right) = \sin\dfrac{\pi}{2} . \cos x + \cos\dfrac{\pi}{2} . \sin x$

$= 1 . \cos x + 0 . \sin x = \cos x$

(ii) $\cos\left(\dfrac{\pi}{2} + x\right) = \cos\dfrac{\pi}{2} . \cos x - \sin\dfrac{\pi}{2} . \sin x$

$= 0 \times \cos x - 1 \times \sin x = -\sin x$

(iii) $\tan\left(\dfrac{\pi}{2} + x\right) = \dfrac{\sin\left(\dfrac{\pi}{2} + x\right)}{\cos\left(\dfrac{\pi}{2} + x\right)} = \dfrac{\cos x}{-\sin x} = -\cot x$

(b) Difference Formulae :

(i) $\sin(A - B) = \sin A . \cos B - \cos A . \sin B$

(ii) $\cos(A - B) = \cos A . \cos B + \sin A . \sin B$

(iii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A . \tan B}$

Proof : Let the reveolving line OM make an angle A with OX and then resolve back to make $\angle MON = B$ so that $\angle XON = A - B$. ***(Fig. 8)***

Let 'P' be any point on ON. Draw $PR \perp OX$, $PT \perp OM$, $TS \perp OX$, $TQ \perp RP$ produced to Q.

Then $\angle TPQ = 90° - \angle PTQ = \angle QTM = A$

Now from Δ OPR, we have

(i) $\text{Sin}(A - B) = \dfrac{PR}{OP} = \dfrac{QR - QP}{OP} = \dfrac{TS - QP}{OP}$

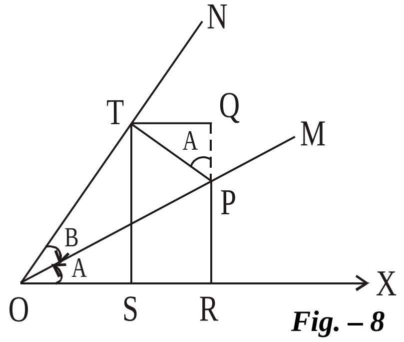


Fig. – 8

$$= \frac{TS}{OP} - \frac{QP}{OP}$$

$$= \frac{TS}{OT} \cdot \frac{OT}{OP} - \frac{OP}{PT} \cdot \frac{PT}{OP}$$

= sin A . cos B – cos A . sin B

(ii) $\cos(A - B) = \frac{OR}{OP} = \frac{OS + SR}{OP} = \frac{OS + TQ}{OP} = \frac{OS}{OP} + \frac{TQ}{QP}$

$$= \frac{OS}{OT} \cdot \frac{OT}{OP} + \frac{TQ}{PT} \cdot \frac{PT}{OP}$$

= cos A . cos B + sin A . sin B

(iii) $\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A . \cos B - \cos A . \sin B}{\cos A \cos B + \sin A \sin B}$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Dividing the numerator and the denominator by cos A. cos B.

(iv) $\cot(A - B) = \frac{\cos(A - B)}{\sin(A - B)}$

$$= \frac{\cos A . \cos B + \sin A . \sin B}{\sin A . \cos B - \cos A . \sin B}$$

$$= \frac{\cot A . \cot B + 1}{\cot B - \cot A}$$

dividing the numerator and denominator by sin A. sin B

We can also deduce substraction formulae from addition formulae in the following manner.

sin(A – B) = sin[A + (–B)]

= sin A . cos (–B) + cos A . sin (–B)

= sin A . cos B + cos A . sin B

cos(A – B) = cos[A + (–B)]

= cos A . cos (–B) – sin A . sin (–B)

= cos A . cos B + sin A . sin B

$\tan(A - B) = \tan[A + (-B)] = \frac{\tan A + \tan(-B)}{1 - \tan A . \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$

Example – 1 : **Find the value of tan 75° and hence prove that tan 75° + cot 75° = 4**

Solution: $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1 \times 1}{\sqrt{3}}} = \frac{\frac{\sqrt{3} + 1}{\sqrt{3}}}{\frac{\sqrt{3} - 1}{\sqrt{3}}}$$

$\therefore \quad \tan 75° = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$

$\therefore$ $\cot 75° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ $\left(\text{since } \cot\theta = \dfrac{1}{\tan\theta}\right)$

$\tan 75° + \cot 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} + \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$

$= \dfrac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{3-1}$ [since $(a + b)(a - b) = a^2 - b^2$]

$\therefore$ $\tan 75° + \cot 75° = 4$

Example – 2 : **If $\sin A = \dfrac{1}{\sqrt{10}}$ and $\sin B = \dfrac{1}{\sqrt{5}}$ show that $A + B = \dfrac{\pi}{4}$**

Solution: $\sin A = \dfrac{1}{\sqrt{10}}$

$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\dfrac{1}{10}} = \sqrt{\dfrac{10-1}{10}} = \sqrt{\dfrac{9}{10}}$

$\therefore$ $\cos A = \dfrac{3}{\sqrt{10}}$

$\sin B = \dfrac{1}{\sqrt{5}}$, $\cos B = \sqrt{1-\sin^2 B}$

$= \sqrt{1-\dfrac{1}{5}} = \sqrt{\dfrac{5-1}{5}} = \sqrt{\dfrac{4}{5}}$

$\therefore$ $\cos B = \dfrac{2}{\sqrt{5}}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$= \dfrac{1}{\sqrt{10}} \times \dfrac{2}{\sqrt{5}} + \dfrac{3}{\sqrt{10}} \times \dfrac{1}{\sqrt{5}} = \dfrac{2}{\sqrt{50}} + \dfrac{3}{\sqrt{50}}$

$= \dfrac{2+3}{\sqrt{50}} = \dfrac{2+3}{5\sqrt{2}}$

$\therefore$ $\sin(A + B) = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\sin(A + B) = \sin 45°$

$\therefore$ $A + B = 45° = \dfrac{\pi}{4}$ $\left[\text{since } 45° = \dfrac{180°}{4}\right]$

Transformation of Sums or Difference in to Products

(a) We have that

$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$ ....(1)

$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$ ....(2)

$\cos(A + B) - \cos(A - B) = 2 \cos A \cos B$ ....(3)

$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B$ ....(4)

Let $A + B = C$ and $A - B = D$

Then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$

Putting the value in formula (1), (2), (3) and (4) we get

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \qquad \text{.... (i)}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \qquad \text{.... (ii)}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \qquad \text{.... (iii)}$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2} \qquad \text{.... (iv)}$$

for practice it is more convenient to quote the formulae verbally as follows :

Sum of two sines = 2 sin (half sum) cos (half difference)

Difference of two sines = 2 cos (half sum) sin (half difference)

Sum of two cosines = 2 cos (half sum) cos (half difference)

Difference of two cosines = 2 sin (half sum) sin (half difference reversed)

[The student should carefully notice that the second factor of the right hand member of IV is sin $\frac{D-C}{2}$, not sin $\frac{C-D}{2}$]

(b) To find the Trigonometrical ratios of Angle 2A in terms of those of A : sin 2A, cos 2A.

Since sin (A + B) = sin A cos B + cos A sin B

putting B = A

sin (A + A) = sin A cos A + cos A sin A

$\Rightarrow$ sin 2A = 2 sin A cos A ..... (i)

cos (A + B) = cos A cos B – sin A sin B

$\Rightarrow$ cos (A + A) = cos A cos A – sin A sin A

$\Rightarrow$ cos 2A = $\cos^2A - \sin^2A$ ..... (ii)

Also cos 2A = $1 - \sin^2A - \sin^2A = 1 - 2\sin^2A$ ..... (iii)

So $2\sin^2A = 1 - \cos 2A$ ..... (iv)

Also cos 2A = $\cos^2A - (1 - \cos^2A) = 2\cos^2A - 1$ ..... (v)

or $2\cos^2A = 1 + \cos 2A$ ..... (vi)

(c) Formula for tan 2A

since $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

Note this formula is not defined when $\tan^2A = 1$ i.e, tan A = ± 1

(d) To express sin 2A and cos 2A in terms of tan A

sin 2A = 2 sin A cos A

$$= 2\frac{\frac{\sin A}{\cos A}}{\frac{1}{\cos^2 A}} = \frac{2\tan A}{\sec^2 A} = \frac{2\tan A}{1 - \tan^2 A}$$

Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(dividing numerator and denominator by $\cos^2 A$)

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(e) To find the Trigonometrical formulae of 3A

$\sin 3A = \sin (2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2 \sin A \cos A . \cos A + (1 - 2 \sin^2 A) \sin A$

$= 2 \sin A(1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A$

$= 3 \sin A - 4 \sin^3 A$

Again, $\cos 3A = \cos (2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A. \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \cos A (1 - \cos^2 A)$

$= 4 \cos^3 A - 3 \cos A$

Also $\tan 3A = \tan (2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} . \tan A}$$

$$= \frac{2 \tan A + \tan A(1 - \tan^2 A)}{1 - \tan^2 A - 2 \tan^2 A}$$

$= \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$, provided $3 \tan^2 A \neq 1$ i.e, $\tan A \neq \pm \dfrac{1}{\sqrt{3}}$

(f) Submultiple Angles :

To express trigonometric ratios of A in terms of ratios of A/2

$\sin 2\theta = 2 \sin \theta \cos \theta$ (true for all value of θ)

Let $2\theta = A$ i.e. $\theta = \dfrac{A}{2}$

$\sin A = 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$ .... (i)

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

or $\cos A = \cos^2 \dfrac{A}{2} - \sin^2 \dfrac{A}{2}$ ..... (ii)

$\cos A = 2 \cos^2 \dfrac{A}{2} - 1 = 1 - 2 \sin^2 \dfrac{A}{2}$ ..... (iii)

Also, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \qquad \text{..... (iv)}$$

[Where $A \neq n\pi + \dfrac{\pi}{2}$, ($n \in I$) and $A \neq (2n+1)\pi$]

Again, $\sin A = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1} = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

[dividing numerator and denomenator by $\cos^2\dfrac{A}{2}$)

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

[where $A \neq (2n+1)\pi$, $n \in I$]

Similarly, $\cos A = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{1} = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

Now dividing numerator and denominator by $\cos^2\dfrac{A}{2}$

$\Rightarrow \quad \cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$ [where $A \neq (2n+1)\pi$, $n \in I$].

Example –1 : Find the values of

(i) $\cos\ 22\frac{1}{2}^{o}$ (ii) $\sin 15^{o}$

Solution : **(i)** We have c $\cos\dfrac{A}{2} = \sqrt{\dfrac{1+\cos A}{2}}$, putting A = 45º

$$\cos 22\frac{1}{2}^{o} = \sqrt{\frac{1+\cos 45^{o}}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(ii) $\sin 15^{o} = \sin(45^{o}-30^{o})$

$= \sin 45^{o}.\cos 30^{o} - \cos 45^{o}.\sin 30^{o}$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example – 2: Prove that sinA.sin (60º – A).sin(60º + A) = $\frac{1}{4}$ sin3A

Solution : sinA.sin (60º – A) sin (60º + A)

= sin A. ($\sin^2 60^\circ - \sin^2 A$) [$\because$ sin (A + B). sin (A – B) = $\sin^2 A - \sin^2 B$]

$$= \sin A\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A\right] = \sin\left[\frac{3}{4} - \sin^2 A\right] = \frac{1}{4} \qquad [3\sin A - 4\sin^3 A]$$

$$= \frac{1}{4}\sin 3A$$

Example – 3: Prove that sin20º.sin40º.sin60º.sin80º = $\frac{3}{16}$

Solution : sin 60º. sin 20º.sin40º. sin 80º

$$= \frac{\sqrt{3}}{2}\text{[sin A. sin (60º – A). sin (60º +A)] where A = 20º}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{4}\cdot\sin 3A = \frac{\sqrt{3}}{8}\cdot\sin 60 = \frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{2} = \frac{3}{16}$$

Example – 4: If A + B + C = π and cos A = cos B . cos C show that tan B + tan C = tan A

Solution : L.H.S. = tan B + tan C

$$= \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin B . \cos C + \cos B . \sin C}{\cos B . \cos C}$$

$$= \frac{\sin (B+C)}{\cos B . \cos C} = \frac{\sin (\pi - A)}{\cos B . \cos C} = \frac{\sin A}{\cos A} = \tan A = \text{R.H.S. (Proved)}$$

Examples – 5: Prove the followings

(a) $\mathbf{\cot 7\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2}$

(b) $\mathbf{\tan 37\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2}$

Solution : **(a)** We know cot $\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}$ (Choosing θ = 15)

$$= \cot 7\frac{1}{2}^\circ = \frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2+\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{2\sqrt{6}+2\sqrt{2}+\sqrt{3}+\sqrt{3}+1+3}{3-1}$$

$$= \frac{2\sqrt{6}+2\sqrt{3}+2\sqrt{2}+4}{2} = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$$

(b) We know $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ (Choosing θ = 75º)

$$\tan 37\frac{1^\circ}{2} = \frac{1-\cos 75^\circ}{\sin 75^\circ} = \frac{1-\cos(90^\circ-15^\circ)}{\sin(90^\circ-15^\circ)}$$

$$= \frac{1-\sin 15^\circ}{\cos 15^\circ} = \frac{1-\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$$

Example – 6: If sin A = K sin B, prove that $\tan\frac{1}{2}$ (A – B) = $\frac{K-1}{K+1}$ tan $\frac{1}{2}$ (A + B)

Solution : Given sin A = K sin B

$$\Rightarrow \frac{\sin A}{\sin B} = \frac{K}{1}$$

Using componendo & dividendo

$$\frac{\sin A+\sin B}{\sin A-\sin B} = \frac{K+1}{K-1}$$

$$\Rightarrow \frac{2\sin\frac{A+B}{2}\,.\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\,.\sin\frac{A-B}{2}} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2}\,.\cot\frac{A-B}{2} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2} = \frac{K+1}{K-1}\,.\tan\frac{A-B}{2}$$

$$\Rightarrow \tan\frac{A-B}{2} = \frac{K-1}{K+1}\tan\frac{A+B}{2}$$

∴ L.H.S. = R.H.S. (Proved)

Example – 7: If (1 – e) $\tan^2\frac{\beta}{2}$ = (1 + e) $\tan^2\frac{\alpha}{2}$, Prove that $\cos\beta = \frac{\cos\alpha - e}{1-e\cos\alpha}$

Solution : (1 – e) $\tan^2\frac{\beta}{2}$ = (1 + e) $\tan^2\frac{\alpha}{2}$ (Given)

$$\tan^2\frac{\beta}{2} = \frac{1+e}{1-e}\tan^2\frac{\alpha}{2}$$

L.H.S = cos β

$$= \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}$$

$$= \frac{1 - e - \tan^2\frac{\alpha}{2} - e\tan^2\frac{\alpha}{2}}{1 - e + \tan^2\frac{\alpha}{2} + e\tan^2\frac{\alpha}{2}} = \frac{\left(1 - \tan^2\frac{\alpha}{2}\right) - e\left(1 + \tan^2\frac{\alpha}{2}\right)}{\left(1 + \tan^2\frac{\alpha}{2}\right) - e\left(1 - \tan^2\frac{\alpha}{2}\right)}$$

$$= \frac{\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}{\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}$$

$$= \frac{\cos\alpha - e}{1 - e\cos\alpha} = \text{R.H.S (Proved)}$$

Example – 8: If A + B + C = π, then Prove the following

(i) sin 2A + sin 2B + sin 2C = 4 sin A. sin B. sin C

Solution : L.H.S. = sin 2A + sin 2B + sin 2C

= 2 sin (A + B) . cos (A – B) + 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) + 2 sin C . cos C

= 2 sin C . cos (A – B) + 2 sin C. cos C

= 2 sin C [cos (A – B) + cos C]

= 2 sin C [cos (A – B) – cos (A + B)]

= 2 sin C . 2 sin A . sin B

= 4 sin A . sin B . sin C R.H.S. (Proved)

(ii) sin 2A + sin 2B – sin 2C = 4 cos A . cos B. sin C

Solution : L.H.S. = sin 2A + sin 2B – sin 2C

= 2 sin (A + B) . cos (A – B) – 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) – 2 sin C. cos C

= 2 sin C. cos (A – B) – 2 sin C. cos C

= 2 sin C [cos (A – B) – cos {π – (A + B)}]

= 2 sin C {cos (A – B) + cos (A + B)}

$$= 2\sin C\left\{2\cos\frac{A - B + A + B}{2} \cdot \cos\frac{A - B - A - B}{2}\right\}$$

= 4 sin C. cos A. cos B.

(iii) sin A + sin B – sin C = 4 sin $\frac{A}{2}$ sin $\frac{B}{2}$ cos $\frac{C}{2}$

Solution : L.H.S. = sin A + sin B – sin C

$$= 2\sin\frac{A + B}{2} \cdot \cos\frac{A - B}{2} - 2\sin\frac{C}{2} \cdot \cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2} \cdot \cos\frac{A - B}{2} - 2\sin\frac{C}{2} \cdot \cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\frac{C}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{(-2)\sin\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right) \cdot \sin\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right\}$$

$$= -4\cos\frac{C}{2} \cdot \sin\frac{A}{2} \cdot \sin\left(-\frac{B}{2}\right)$$

$$= 4\cos\frac{C}{2} \cdot \sin\frac{A}{2} \cdot \sin\frac{B}{2} = \text{R.H.S (Proved)}$$

ASSIGNMENT

1. If $\tan\alpha = \frac{1}{2}, \tan\beta = \frac{1}{3}$, then find the value of $(\alpha + \beta)$

2. Find the value of $\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ}$

3. Prove that $\frac{1}{\tan 3A - \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 2A$

4. If A + B = 45º, show that (1 + tanA) (1+ tanB) = 2

5. If $(1 - e)\ \tan^2\frac{\beta}{2} = (1+e)\ \tan^2\frac{\alpha}{2}$

Prove that $\cos\beta = \frac{\cos\alpha - e}{1 - e\cos\alpha}$

6. If A + B + C = π, prove that
cos2A + cos2B + cos 2C + 1 + 4 cos A.cosB.cosC = 0

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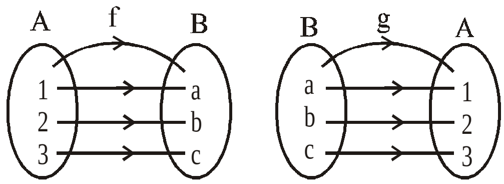
CHAPTER – 9

INVERSE TRIGONOMETRIC FUNCTIONS

INVERSE FUNCTION :

If f : A → B be a bijective function or one to one onto function from set A to the set B. As the function is 1 – 1, every element of A is associated with a unique element of B. As the function is onto, there is no element of B which in not associated with any element of A. Now if we consider a function g from B to A, we have for f ∈ B there is unique x ∈ A. This g is called inverse function of f and is denoted by f^{-1}.



INVERSE TRIGONOMETRIC FUNCTION :

We know the equation x = siny means that y is the angle whose sine value is x then we have $y = \sin^{-1}x$ similarly $y = \cos^{-1}x$ if x = cosy and $y = \tan^{-1}x$ is x = tan y etc.

The function $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$, $\cot^{-1}x$ are called inverse trigonometric function.

* Properties of inverse trigonometric function.

I. Self adjusting property :

(i) $\sin^{-1}(\sin\theta) = \theta$

(ii) $\cos^{-1}(\cos\theta) = \theta$

(iii) $\tan^{-1}(\tan\theta) = \theta$

Proof:

(i) Let $\sin\theta = x$, then $\theta = \sin^{-1}x$

$\therefore \sin^{-1}(\sin\theta) = \sin^{-1}x = \theta$

proofs of (ii) * (iii) as above.

II. Reciprocal Property :

(i) $\text{cosec}^{-1}\frac{1}{x} = \sin^{-1}x$

(ii) $\sec^{-1}\frac{1}{x} = \cos^{-1}x$

(iii) $\cot^{-1}\frac{1}{x} = \tan^{-1}x$

Proof :

(i) Let x = sinθ then cosecθ = $\frac{1}{x}$

so that θ = $\sin^{-1}x$ & θ = $\text{cosec}^{-1}\frac{1}{x}$

∴ $\sin^{-1}x = \text{cosec}^{-1}\frac{1}{x}$

(ii) and (iii) may be proved similarly

III. Conversion property :

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

(ii) $\cos^{-1}x \ \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

Proof:

(i) Let θ = $\sin^{-1}x$ so that sin θ = x

Now $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

i.e., $\theta = \cos^{-1}\sqrt{1-x^2}$

Also $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Thus we have $\theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Theorem – 1 : Prove that

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

Proof :

(i) Let $\sin^{-1}x$ = θ, then

x = sin θ = $\cos\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \cos^{-1}x = \frac{\pi}{2}-\theta = \frac{\pi}{2} - \sin^{-1}x$

$\Rightarrow \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) and (iii) can be proved similarly.

Theorem – 2 : If xy < 1, then

$$\mathbf{\tan^{-1}x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

Proof : Let $\tan^{-1}x = \theta_1$ and $\tan^{-1} y = \theta_2$

Then

$\tan \theta_1 = x$ and $\tan \theta_2 = y$

$$\Rightarrow \quad \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1-\tan\theta_1 \tan\theta_2} = \frac{x+y}{1-xy}$$

$$\Rightarrow \quad \theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Theorem – 3 : $\mathbf{\tan^{-1}x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)}$

Proof : Let $\tan^{-1}x = \theta$, and $\tan^{-1}y = \theta_2$

$$\Rightarrow \quad \tan\theta_1 = x \text{ and } \tan\theta_2 = y$$

$$\Rightarrow \quad \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1, \tan\theta_2} = \frac{x-y}{1+xy}$$

$$\Rightarrow \quad \theta_1 - \theta_2 = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \quad \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

Note : $\tan^{-1} + \tan^{-1} y + \tan^{-1}z$

$$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Theorem – 4 : Prove that :

(i) $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

Proof :

(i) Let $\sin^{-1}x = \theta$, Then, $x = \sin\theta$

$$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta.\ \sqrt{1-\sin^2\theta}$$

$$= 2x\sqrt{1-x^2}$$

$$\Rightarrow \quad 2\theta = \sin^{-1}\left[2x\sqrt{1-x^2}\right] \quad \Rightarrow \quad 2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$$

(ii) Let $\cos^{-1}x = \theta$ Then, $x = \cos\theta$

$$\therefore \quad \cos 2\theta = (2\cos^2\theta - 1) = 2x^2 - 1$$

$$\Rightarrow \quad 2\theta = \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow \quad 2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$$

Theorem – 5 : Prove that

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{(1-x^2)(1-y^2)}\right]$

(iii) $\sin^{-1}x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{(1-x^2)(1-y^2)}\right]$

Proof :

(i) Let $\sin^{-1}x = \theta_1$, and $\sin^{-1}y = \theta_2$, Then

$\sin \theta_1 = x$ and $\sin \theta_2 = y$

$\therefore$ $\sin (\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$

$= \sin \theta_1 \sqrt{1-\sin^2\theta_2} + \sqrt{(1-\sin^2\theta_1)} \sin\theta_2$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow$ $\theta_1 + \theta_2 = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

$\Rightarrow$ $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

The other results may be proved similarly.

Example – 1 : **If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$**

Solution : Given $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = (\pi - \cos^{-1}z)$

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos(\pi - \cos^{-1}z)$

$\Rightarrow$ $xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\cos(\cos^{-1}z) = -z$

$\Rightarrow$ $xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow$ $(xy + z)^2 = (1 - x^2)(1 - y^2) = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow$ $x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2xyz = 1$ (Proved)

Example – 2 : **Find the value of $\cos\tan^{-1}\cot\cos^{-1}\frac{\sqrt{3}}{2}$**

Solution : $\cos^{-1}\frac{\sqrt{3}}{2} = \theta \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\theta = \frac{\pi}{6} \Rightarrow \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\therefore$ $\cos\tan^{-1}\cot\cos^{-1}\frac{\sqrt{3}}{2} = \cos\tan^{-1}\cot\frac{\pi}{6}$

$= \cos\tan^{-1}\sqrt{3}\left[\because \tan^{-1}\sqrt{3} = \frac{\pi}{3}\right] = \cos\frac{\pi}{3} = \frac{1}{2}$

Example – 3 : Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$.

Solution : L.H.S $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} \qquad \left(\because 2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2}\right)$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{1}{2}+\frac{1}{7}}{1-\frac{1}{2}\times\frac{1}{7}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{9}{14}}{\frac{13}{14}} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{9}{13}$$

$$= \tan^{-1}\frac{\frac{1}{2}+\frac{9}{13}}{1-\frac{1}{2}\times\frac{9}{13}} = \tan^{-1}\frac{\frac{31}{26}}{\frac{17}{26}} = \tan^{-1}\frac{31}{17} = \text{R.H.S. (Proved)}$$

Example – 4 : Prove that $\cot^{-1}9 + \text{cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$

Solution : L.H.S. $= \cot^{-1}9 + \text{cosec}^{-1}\frac{\sqrt{41}}{4}$

$$= \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} \qquad \left[\because \text{cosec}^{-1}\frac{\sqrt{41}}{4} = \tan^{-1}\frac{4}{5}\right]$$

$$= \tan^{-1}\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}.\frac{4}{5}} = \tan^{-1}\frac{\frac{5+36}{45}}{\frac{45-4}{45}} = \tan^{-1}\frac{\frac{41}{45}}{\frac{41}{45}}$$

$$= \tan^{-1}1 = \frac{\pi}{4} \text{ R.H.S. (Proved)}$$

ASSIGNMENT

1. Find the value of $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

2. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$. Show that

$$x.\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

3. If s $\sin^{-1}\frac{x}{5} + \text{cos}ec^{-1}\frac{5}{4} = \frac{\pi}{2}$. Find the value of x.

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CHAPTER – 10

PROPERTIES OF TRIANGLE

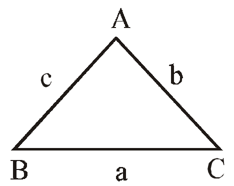
Introduction :

In any triangle ABC, the angles of a triangle are deroted by the capital letters A, B, C. The sides BC, CA and AB opposite to the angles A, B, C are respectively denoted by a, b & c. The six elements are not independent but connected by the relation.

(i) $A + B + C = \pi$

(ii) $S = a + b + c$

(iii) $a + b + c$; $b + c > a$; $c + a > b$



In addition to these relations, the elements are of triangle are connected by some trigorometric relations.

Sine formula :

In any triangle ABC, the sides are proportional to the sines of the opposite angles, i.e,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Case – I : Let the triangle ABC be acute angled. AD is perpendicular to BC ***(Fig.1)***.

$\sin B = \frac{AD}{c}$, AD = b sin C; similarly AD = c sin B

$\therefore$ b sin C = c sin B

$$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{...... (1)}$$

In the similar manner, we can prove that

$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad \text{....... (2)}$$

from eqution (1) and (2)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

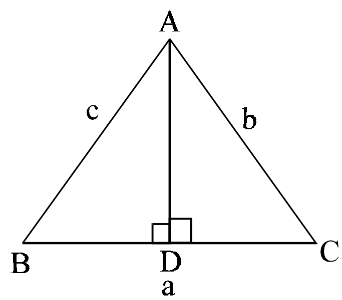


Fig. – 1

Case –II : Let the triangle ABC be right angled triangle ***(Fig.2)***,

$\angle C = 90°$

$\sin C = 1$

$\sin A = \frac{a}{c}$, $\sin B = \frac{b}{c}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1} = \frac{c}{\sin C}$$

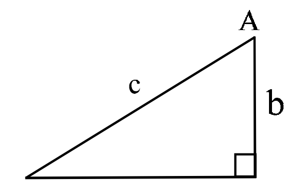


Fig. – 2

Case –III : Let the triangle ABC be an obtuse angle such that $\angle C > 90°$ ***(Fig.3)***

In Δ ABD, $\sin B = \dfrac{AD}{c}$

$AD = c \sin B$

In Δ ACD, $\sin(\pi - c) = \dfrac{AD}{b}$

$AD = b \sin C$

$b \sin C = c \sin B$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Hence proved

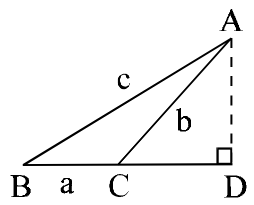


Fig. – 3

Cosine Formula :

In any triangle ABC to find the cosine of an angle in terms of the sides

(i) Let A be an acute angle, draw BD perpendicular to AC ***(Fig.4)***.

Then geometry we have

$BC^2 = AB^2 + AC^2 - 2AC \cdot AD$

$= AB^2 + AC^2 - 2AC \cdot AB \cdot \dfrac{AD}{AB}$

or $a^2 = b^2 + c^2 - 2bc \cos A$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

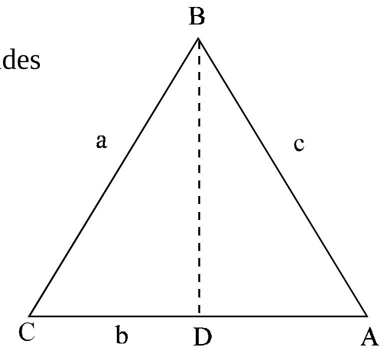


Fig. – 4

(ii) Let A be an obtuse angle, Draw perpendicular BD from B to CA produced.

Then from geometry, we have ***(Fig.5)***

$BC^2 = AB^2 + AC^2 + 2AC \cdot AD$

or $a^2 = b^2 + c^2 + 2\,b.c.\cos(180° - A)$

or $a^2 = b^2 + c^2 - 2bc \cos A$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

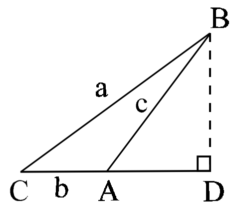


Fig. – 5

(iii) Let A be right angle ***(Fig. 6)***

Then from geometry,

$a^2 = b^2 + c^2$ as $\angle A = 90°$

$a^2 = b^2 + c^2 - 2bc \cos A$

$\therefore \cos A = \cos 90° = 0$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

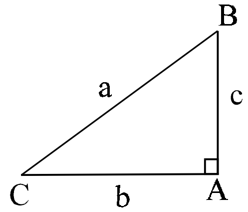


Fig. – 6

Similarly it may be shown that

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \text{and} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

(C) Projection formula :

In any triangle ABC, rieve there

In (Fig.7 (i)) $a = b \cos C + c \cos B$

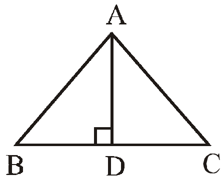


Fig.7(i)

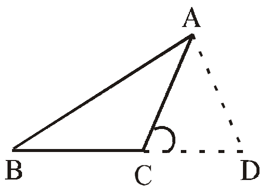


Fig.7 (ii)

$BC = BD + DC$ or $a = c \cos B + b \cos C$

In Fig.7 (ii)

$BC = BC - CD$

But $\dfrac{BD}{BA} = \cos B$ or $BD = BA \cos B = C \cos B$

and $\dfrac{CD}{CA} = \cos(180^o - C) = -\cos C$

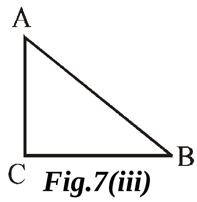


Fig.7(iii)

$BC = BD - CD$

$a = c \cos B - (-b \cos C) = C \cos B + b \cos C$

Fig.7(iii)

$BC = BD$

But $\dfrac{BD}{BA} = \cos B$

or $BD = BA \cos B = C \cos B$

$a = C \cos B + 0 = C \cos B + b \cos c$ ($\therefore$ $\cos C = \cos 90^o = 0$)

Similarly $b = C \cos A + a \cos C$

$C = a \cos B + b \cos A$

Area of a triangle (Herons formula) ;

The area of a triangle in given by $= \sqrt{s(s-a)\,(s-b)\,(s-c)}$

where $2s = a + b + c$ is the perimeter of the triangle.

More Formulae

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}, \sin\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{ac}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s.(s-a)}{bc}}, \cos\frac{B}{2} = \sqrt{\frac{s.(s-b)}{ca}} \quad \cos\frac{C}{2} = \sqrt{\frac{s.(s-c)}{ab}}$$

Area of a triangle in terms of sides (Heron's Formula)

$$\Delta = \frac{1}{2}\text{ bc sin A} = \frac{1}{2}\text{ bc . 2 sin }\frac{A}{2}\cos\frac{A}{2}$$

$$= bc\sqrt{\frac{(s-b)(b-c)}{bc}}\quad\sqrt{\frac{s(s-a)}{bc}}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Example –1 : In any triangle ABC show that

$$a^2(\sin^2B - \sin^2C) + b^2(\sin^2C - \sin^2A)\ C^2(\sin^2A - \sin^2B) = 0$$

Proof : We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{K}$

$\Rightarrow \sin A = ka, \sin B = kb, \sin C = kc$

Now L.H.S

$(k^2a^2(b^2 - c^2) + k^2b^2(c^2 - a^2) + k^2c^2(a^2 - b^2)$

$= k^2\{a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2)\}$

$= k^2 - 0 = 0$ R.H.S

Example –2 : If A : B : C = 1 : 2 : 3

Then show that sinA : sin B: sin C = 1 : $\sqrt{3}:$ **2**

Proof : A : B: C = 1 : 2 : 3 (Given)

$\Rightarrow A = K, B = 2K, C = 3K$

$$A + B + C = \pi \Rightarrow 6k = \pi \Rightarrow k = \frac{\pi}{6}$$

$$\therefore\ A = \frac{\pi}{6},\ B = \frac{\pi}{3},\ C = \frac{\pi}{2}$$

$$\therefore \sin A : \sin B : \sin C = \sin = \frac{\pi}{6} : \sin = \frac{\pi}{3} : \sin = \frac{\pi}{2}$$

$$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2$$

Example – 3 : In any triangle ABC prove that cos $\mathbf{\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2}}$

Proof: R.H.S $\frac{b+c}{a}\sin\frac{A}{2}$

$$= \frac{k\sin B + k\sin C}{k\sin A} : \sin\frac{A}{2} \qquad \left(\because\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\right)$$

$$= \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\frac{B+C}{2}.\cos\frac{B-C}{2}}{2\sin\frac{A}{2}.\cos\frac{A}{C}}.\sin A$$

$$= \frac{\cos\frac{A}{C}\cdot\cos\frac{B-C}{2}}{\cos\frac{A}{2}} \quad \left(\because\ \sin\frac{B+C}{2} = \cos\frac{A}{2}\right)$$

$$= \cos\frac{B-C}{2}$$

ASSIGNMENTS

1. In any triangle ABC, Prove that $\sum\frac{\cos A}{\sin B.\sin C} = 2$

2. Prove that If (a + b + c) (b + c – a) = 3bc, prove that ∠60º

3. If $\frac{b+c}{5} = \frac{c+a}{6} = \frac{a+b}{7}$ then sinA : sin B : sinC = 4 : 3 : 2

❒ ☸ ❒

CHAPTER – 11

VECTORS

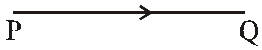
Introduction :

At present vector methods are used in almost all branches of science such as Mechanics, Mathematics, Engineering, physics and so on. Both the theory and complicated problems in these subjects can be discussed in a simple manner with the help of vectors. It is a very useful tool in the hands of scientists.

Physical quantities are divided into two category scalar quantities and vector quantities. Those quantities which have any magnitude and which and not related to any fixed direction scalars. Example of scalars are mass volume density, work, temperature etc. Second kind of quantities are those which have both magnitude and direction. Such quantities are vectors. Displacement, velocity, acceleration, momentum weight, force etc. are examples of vector quantities.

Representation of vectors :

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end indicates the direction of vector. A vector denoted by $\overrightarrow{PQ}$, is determined by two points P, Q such that the direction of the vector is the length of the straight line PQ and its direction is that from P to Q. The point P is called initial point of vector $\overrightarrow{PQ}$ and Q is called terminal points.



Note : The length (magnitude or modulus) of $\overrightarrow{AB}$ or $\vec{a}$ generally denoted my $|\overrightarrow{AB}|$ or $|\vec{a}|$ thus $|\vec{a}|$ = length (magnitude or modulus ? or vector $\vec{a}$)

Types of vectors :

(i) **Zero vector or null vector :** A vector whose initial so terminal points are coincident is called zero or the null vector. The modulus of a null vector is zero.

(ii) **Unit vector :** A vector whose modulus in unity, is called a unit vector. The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$. Thus $|\hat{a}| = 1$

(iii) **Like and unlike vector :** Vectors are said to be like when they have same sense of direction and unlike when they have opposite directions.

(iv) **Collinear or Parallel vector :** Vectors having the same or parallel supports are called collinear vectors.

(v) **Co-initial vectors:** Vectors having the same initial point are called co-initial vector.

(vi) **Co-planner vector :** A system of vector and said to be co-planner in their supports are parallel to the same plane.

(vii) **Negative of a vector :** The vector which has the same Magnitude as the vector $\vec{a}$ but opposite direction, is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. There if $\overrightarrow{PQ} = \vec{a}$ then $\overrightarrow{QP} = -\vec{a}$.

Operations on Vectors

Addition of Vectors :

Tringle Law of Addition of Two Vectors :

The law states that if two vectors are represented by the two sides of a triangle, taken in order, then their sum (or resultant) is represented by the third side of the triangle but in the reverse order.

Let $\vec{a}$, $\vec{b}$ be the given vectors. Let the vector $\vec{a}$ be represented by the directed segment $\overrightarrow{OA}$ and the vector $\vec{b}$ be the directed segment $\overrightarrow{AB}$ so that the terminal point A of $\vec{a}$ is the initial

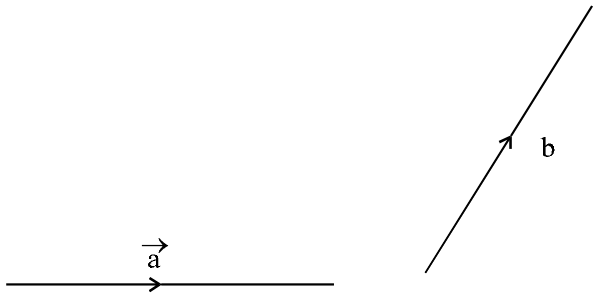


Fig. 1

point of $\vec{b}$. Then the directed segment OB (i.e. $\overrightarrow{OB}$) represents the sum (or resultant) or $\vec{a}$ and $\vec{b}$ and is written as $\vec{a} + \vec{b}$ ***(fig. 2)***

Thus, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

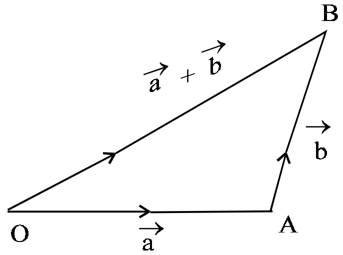


Fig.2

Note : 1. The method of drawing a triangle in order to define the vector sum ($\vec{a} + \vec{b}$) is called **triangle law of addition of the vectors.**

2. Since any side of a triangle is less than the sum of the other two sides.

$\therefore$ Modulus of $\overrightarrow{OB}$ is not equal to the sum of the modulus of $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

Parallelogram Law of Vectors

If two vectors $\vec{a}$ and $\vec{b}$ are represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common initial point.

Let $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Then

$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ ***.(fig.3)***

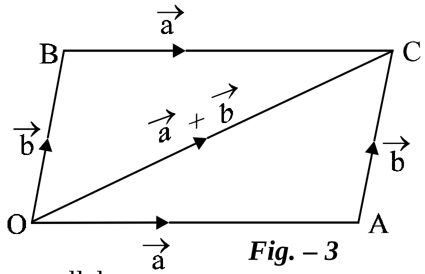


Fig. – 3

i.e. Their sum $\vec{a} + \vec{b}$ is respesented by the diagonal $\overrightarrow{OC}$ of the parallelogram.

Polygon Law of addition of Vectors

To add n vectors $\vec{a}_1$, $\vec{a}_2$, ... $\vec{a}_n$ we choose O as an origin ***(fig.4)*** and draw.

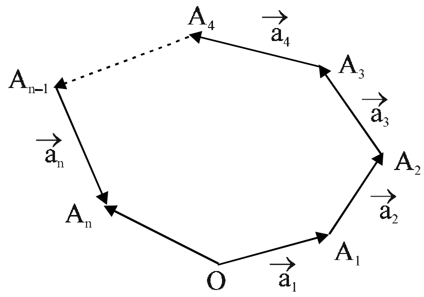


Fig. – 4

$\overrightarrow{OA_1} = \vec{a}_1,\ \overrightarrow{A_1A_2} = \vec{a}_2, ...\overrightarrow{A_{n-1}A_n} = \vec{a}_n$

$\therefore\ \vec{a}_1 + \vec{a}_2 + ... + \vec{a}_n$

$= \overrightarrow{OA_1} + \overrightarrow{A_1A_2} + ... + \overrightarrow{A_{n-1}A_n}$

$= (\overrightarrow{OA_1} + \overrightarrow{A_1A_2}) + \overrightarrow{A_2A_3} + ... + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_2} + \overrightarrow{A_2A_3} + ... + \overrightarrow{A_{n-1}A_n}$

$= \overrightarrow{OA_3} + \overrightarrow{A_3A_4} + ... + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n}$

Hence the sum of vectors $\vec{a}_1$, $\vec{a}_2$ ...... $\vec{a}_n$ is represented by $\overrightarrow{OA_n}$. This method of vector addition is called ‘‘polygon law of addition of vectors.

Corollary : From the polygon law of addition of vectors, we have

$\overrightarrow{OA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + ... + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n} = -\overrightarrow{A_nO}$

$\overrightarrow{OA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + ... + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nO} = \overrightarrow{A_nO}$ (Null vector)

$\therefore$ The sum of vectors determined by the sides of any polygon taken in order is zero.

Properties of Vectors Addition

(1) Vector Addition is Commulative :

If $\vec{a}$ and $\vec{b}$ be any two vectors, then

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Proof : Let the vectors $\vec{a}$ and $\vec{b}$ be represented by the directed segments $\overrightarrow{OA}$ and $\overrightarrow{AB}$ respectively so that ***(fig.4)***

$\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{AB}$

Now $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} \Rightarrow \overrightarrow{OB} = \vec{a} + \vec{b}$

Complete the || gm OABC

Then $\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$ and $\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$

$\therefore\ \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \vec{b} + \vec{a}$

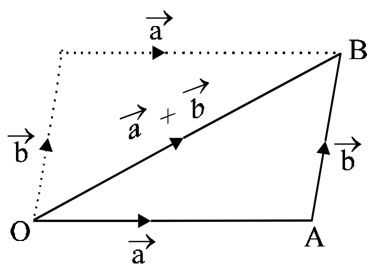


Fig.– 4

From (1) and (2) , we have

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

2. Vector Addition is Associative.

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors, then $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.

Proof : Let the vectors $\vec{a}, \vec{b}, \vec{c}$ .be represented by the directed segments $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}$ respectively; so that *(fig.5)*

$\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{BC}$

Then $\vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC})$

$= \overrightarrow{OA} + \overrightarrow{AC}$ [Δ Law of addition]

$= \overrightarrow{OC}$ [Δ Law of addition]

$\therefore\ \vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{OC}$ ....(1)

Again, $(\vec{a} + \vec{b}) + \vec{c} = (\overrightarrow{OA} + \overrightarrow{AB}) + \overrightarrow{BC}$

$= \overrightarrow{OB} + \overrightarrow{BC}$ [Δ Law of addition]

$= \overrightarrow{OC}$ [Δ Law of addition]

$\therefore\ (\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{OC}$ ....(2)

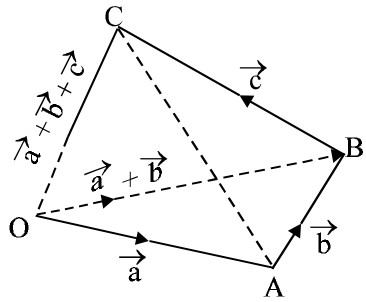


Fig. – 5

From (1) and (2), we get

$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

Remarks : The sum of three vectors $\vec{a}, \vec{b}, \vec{c}$ is independent of the order in which they are added and is written as $\vec{a} + \vec{b} + \vec{c}$.

(3) Existence of Additive Identity :

For any vector $\vec{a}$, $\vec{a} + \vec{O} = \vec{a}$, where $\vec{O}$ is a null (zero) vector.

Proof : Let the vector $\vec{a}$ be represented by the directed segment $\overrightarrow{OA}$; so that $\vec{a} = \overrightarrow{OA}$.

Also let the Zero Vector $\vec{O}$ be represented by the directed segment $\overrightarrow{AA}$;

So that $\vec{O} = \overrightarrow{AA}$

Then $\vec{a} + \vec{O} = \overrightarrow{OA} + \overrightarrow{AA}$

$= \overrightarrow{OA}$ [By Triangle law of addition]

$= \vec{a}$

Thus, $\vec{a} + \vec{O} = \vec{a}$

Note : In view of the above property, the null vector is called the additive identity.

Property 4 : Existence of Additive Inverse

For any vector $\vec{a}$, there exists another vector $-\vec{a}$ such that

$$\vec{a} + (-\vec{a}) = \vec{O}$$

Proof : Let $\overrightarrow{OA} = \vec{a}$, there exists another $\overrightarrow{AO} = -\vec{a}$

$\therefore\ \vec{a} + (-\vec{a}) = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = \vec{O}$ [By Δ Law]

Note : In view of the above property, the vector ($-\vec{a}$) is called the **additive inverse** of the vector $\vec{a}$.

Substraction of Vectors :

If $\vec{a}$ and $\vec{b}$ are two given vectors, then the substraction of $\vec{b}$ from $\vec{a}$ (denoted by $\vec{a}-\vec{b}$) is defined as addition of $-\vec{b}$ to $\vec{a}$.

i.e. $\vec{a}-\vec{b} = \vec{a} + (-\vec{b})$

$\therefore$ It is clear that

Multiplication of a Vector by a Scalar

If $\vec{a}$ is any given vector and m is any given scalar, then the product m $\vec{a}$ or $\vec{a}$ m of the vector $\vec{a}$ and the scalar m is a vector whose

(i) **Magnitude =** |m| times that of the vector $\vec{a}$.

In other words, $m\vec{a} = |m| \times |\vec{a}|$

$= m \times |\vec{a}|$ if $m \geq 0$

$= -m \times |\vec{a}|$ if $m < 0$

(ii) Support is same or parallel to that of the support of $\vec{a}$

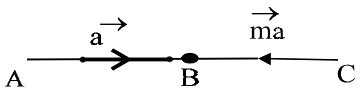
and (iii) Sense is same to that of $\vec{a}$ if $m > 0$ and opposite to that of $\vec{a}$ if $m < 0$.

Geometrical Representation :

Let the vector $\vec{a}$ be represented by the directed segment $\overrightarrow{AB}$

Case I . Let m > 0. Choose a point C and AB on the same side of A as B such that

$|\overrightarrow{AC}| = m\,|\overrightarrow{AB}|$, ***(fig.6)***



(m > 0)

Fig. – 6

Then the vector $\overrightarrow{ma}$ is represented by $\overrightarrow{AC}$.

Case II : Let m<0. Choose a point C on AB on the side of A opposite so that of B such that, ***(fig.7)***

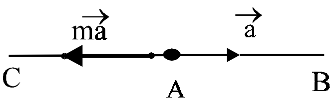


Fig. –7

$$|\overrightarrow{AC}| = m\,, |\overrightarrow{AB}|$$

Then the vector $\overrightarrow{ma}$ is represented by $\overrightarrow{AC}$.

Linearly Dependent and Independent Vectors

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be **linearly dependent** if there exists a scalar t (≠0), such that $\vec{a} = t\,\vec{b}$

This can be the case if and only if the vectors $\vec{a}$ and $\vec{b}$ are parallel.

If the vectors $\vec{a}$ and $\vec{b}$ are not **linear dependent** they are said to be linearly independent and in this case $\vec{a}$ and $\vec{b}$ are not parallel vectors.

Thus, if $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$, then $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if A, B, C lie in a straight line; othewise they are linearly independent.

Properties of Multiplication of a Vector by a Scalar

(I) **Associative Law**

If $\vec{a}$ is any vector and m, n are any scalars, then m($\overrightarrow{na}$) = mn ($\vec{a}$)

Proof : If any one or more of m, n or a are zero, then m($\overrightarrow{na}$) = mn ($\vec{a}$)[$\because$ Each side = $\vec{0}$]

While if m ≠ 0, n ≠ 0, $\vec{a} \neq \vec{0}$, then the following four cases arise :

(i) m > 0, n < 0 (ii) m < 0, n > 0

(iii) m > 0, n >0 (iv) m < 0, n < 0

Case (I) : When m > 0, n < 0

Let $\vec{a}$ be represented by the directed line segment $\overrightarrow{AB}$.***(fig .8)***

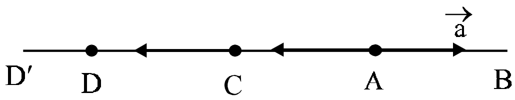


Fig. 8

Since n < 0, take a point C on AB on the side of A opposite to that of B such that $\overrightarrow{AC}$ represents $\overrightarrow{na}$

i.e.| $\overrightarrow{AC}$ | = |n| | $\overrightarrow{AB}$ |

Since m > 0, take point D on AB on the same side of A as C such $\overrightarrow{AD}$ represents m ($\overrightarrow{na}$)

i.e. | $\overrightarrow{AD}$ | = m | $\overrightarrow{AC}$ |=m|n| $\overrightarrow{AB}$ |......[1]

Again, since m > 0, n < 0, so that mn < 0; take a point D′ on AB on the side of A opposite to that of B such that AD′ represents |mn| $\vec{a}$.

i.e.,| $\overrightarrow{AD'}$ | = |mn| | $\overrightarrow{AB}$ | = |m|n|| $\overrightarrow{AB}$ | = m |n| | $\overrightarrow{AB}$ | [$\because$ |m|= m as m > 0]

$\therefore$ | $\overrightarrow{AD'}$ | = m |n| | $\overrightarrow{AB}$ | ......(2)

From (1) and (2), we get

| $\overrightarrow{AD'}$ | = | $\overrightarrow{AD}$ |

which shows that D and D′ coincide, proving that m ($\overrightarrow{na}$) = (mn) $\vec{a}$

Proceeding on the same lines, the other three cases can be similarly proved.

(2) **Distributive Law :** If m, n are any scalars and $\vec{a}$ is any vector, then

(m + n) $\vec{a}$ = $\overrightarrow{ma}$ + $\overrightarrow{na}$

Proof : If $\vec{a}$ = $\vec{0}$ or m, n are both zero, then

(m + n) $\vec{a}$ = $\overrightarrow{ma}$ + $\overrightarrow{na}$ [$\because$ Each side = $\vec{0}$]

But if $\vec{a} \neq \vec{0}$, the following three cases arise :

(1) m + n > 0 (2) m + n = 0 and

(3) m + n < 0

Case - I . Here m + n > 0

The following sub-cases arise :

(i) m > 0, n > 0 (ii) m > 0 , n < 0

and (iii) m < 0, n > 0

Let $\vec{a}$ be represented by the directed segment $\overrightarrow{AB}$.

Since m + n > 0, take a point C on AB on the same side of A as B such that $\overrightarrow{AC}$ represents (m + n) $\vec{a}$. ***(fig. 9)***

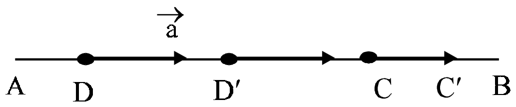


Fig. – 9

i.e. $|\overrightarrow{AC}| = (m + n)\,|\overrightarrow{AB}|$ ....(1)

Sub-case, (i) Since m > 0 , take a point D on the same side of A as B such that $\overrightarrow{AB}$ represent $\overrightarrow{ma}$.

i.e. $|\overrightarrow{AD}| = m\,|\overrightarrow{AB}|$ ....(2)

Again since n > 0, take a point D′ on the same side of A as B such that $\overrightarrow{AD'}$ represents $\overrightarrow{na}$.

i.e. $|\overrightarrow{AD'}| = n\,|\overrightarrow{AB}|$ ....(3)

Thus, $\overrightarrow{ma} + \overrightarrow{na}$ is represented by $\overrightarrow{AC'}$ (where C′ is on the same side of A as B) such that

$|\overrightarrow{AC'}| = |\overrightarrow{AD}| + |\overrightarrow{AD'}|$

$= m\,|\overrightarrow{AB}| + n\,|\overrightarrow{AB}|$ [From (2) and (3)]

$\Rightarrow |\overrightarrow{AC'}| = (m + n)\,|\overrightarrow{AB}|$ ....(4)

$\because$ From (1) and (4), $|\overrightarrow{AC'}| = |\overrightarrow{AC}|$, which shows that C and C′ coincide, proving that

$(m + n)\vec{a} = \overrightarrow{ma} + \overrightarrow{na}$

The other sub-cases of case (1) may be similarly proved.

Proceeding in the same way, we can prove the result for case 2 and case 3 also.
case 2 and 3 also.

3. If $\vec{a}$ and $\vec{b}$ are any two vectors and m is a scalar, then

$m(\vec{a} + \vec{b}) = \overrightarrow{ma} + \overrightarrow{mb}$.***(fig.10)***

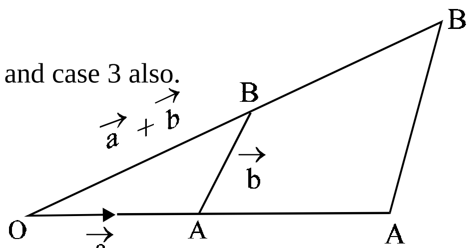


Fig. – 10

Position Vector of a Point

Let O be any point called the origin of reference or simple the origin. Let P be any other point.

Then $\overrightarrow{OP}$ is called the position vector of the point P relative to the point O.

Hence, with the choice of O as the origin of reference, a vector can be associated to every point P and conversely.***(fig .11)***

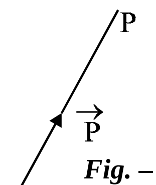


Fig. – 11

Representation of a vector in terms of the position vectors of its end points :

Let A and B be two given points and $\vec{a}$, $\vec{b}$ the position vectors of A, B respectively relative to a point O as the origin of reference; so that ***(fig. 12)***

$\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

$\therefore$ From ΔOAB

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ [By Δ law of addition]

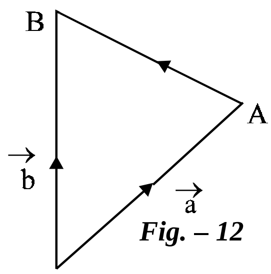


Fig. – 12

$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

Note : $\overrightarrow{AB}$ = Position vector B – Position vector A.

SECTION FORMULA :

Statement . If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the point C which divides AB in the ratio m : n, where m and n are positive real numbers, has the position vector.

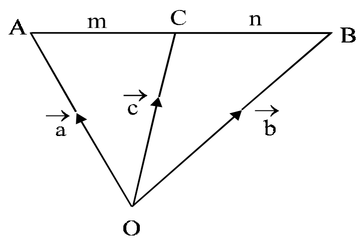
$$\vec{\mathbf{c}} = \frac{\mathbf{n}\vec{\mathbf{a}} + \mathbf{m}\vec{\mathbf{b}}}{\mathbf{m}+\mathbf{n}}$$

Proof : Let O be the origin of reference and let $\vec{a}$ and $\vec{b}$ be the position vectors of the given points A and B so that ***(fig.13)***

$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$

Let C divide AB in the ratio m : n

$\therefore \dfrac{AC}{CB} = \dfrac{m}{n}$ ....(i)



Hence $\dfrac{m}{n}$ is positive or negative according as C divides AB internally or externally.

Fig. – 13

We have to express the position vector $\overrightarrow{OC}$ of the point C in terms of those of A and B.

We re-write (1) as, nAC = mCB.

And obtain the vector equality $n\overrightarrow{AC} = m\overrightarrow{CB}$. Expressing the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of the position vectors of the end points, we obtain

$n(\overrightarrow{OC} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OC})$

$\Rightarrow (m+n)\overrightarrow{OC} = n\overrightarrow{OA} + m\overrightarrow{OB}$

$$\Rightarrow \overrightarrow{OC} = \frac{n\overrightarrow{OA} + m\overrightarrow{OB}}{m+n} = \frac{n\vec{a} + m\vec{b}}{m+n}$$

Mid-point formula : If C is the mid-point of AB. then m : n = 1 : 1

$\therefore$ The position vector of c is given by $\overrightarrow{OC} = \dfrac{1\cdot\vec{a} + 1\cdot\vec{b}}{2}$

i.e. $\overrightarrow{OC} = \dfrac{\vec{a}+\vec{b}}{2}$

Hence the position vector of the mid point of the join of two points with position vectors, $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}+\vec{b}}{2}$ or $\dfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

Example – 1 : Prove that

(i) $|\vec{\mathbf{a}} + \vec{\mathbf{b}}| \le |\vec{\mathbf{a}}| + |\vec{\mathbf{b}}|$ **(ii)** $|\vec{\mathbf{a}}| - |\vec{\mathbf{b}}| \le |\vec{\mathbf{a}} - \vec{\mathbf{b}}|$ **(iii)** $|\vec{\mathbf{a}} - \vec{b}| \le |\vec{\mathbf{a}}| + |\vec{\mathbf{b}}|$

Solution : (i) When A, B, C are not -collinear, draw a ΔABC such that, ***(fig.14)***

$\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$

Then $\vec{a} + \vec{b} = \overrightarrow{AC}$ [By Addition Law]

$\therefore$ AC < AB + BC (As sum of two sides is greater than the third side)

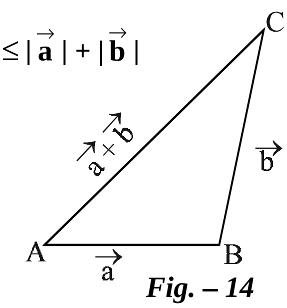


Fig. – 14

$\therefore |\overrightarrow{AC}| < |\overrightarrow{AB}| + |\overrightarrow{BC}|$

$|\vec{a}+\vec{b}| < |\vec{a}| + |\vec{b}|$ ....(1)

$[\because \overrightarrow{AB} = \vec{a}, \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{AC} = \vec{a}+\vec{b}]$

When A, B and C are collinear, then, ***(fig.15)***

$\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{BC}$

$\vec{a}+\vec{b} = \overrightarrow{AC}$

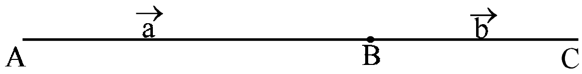


Fig.– 15

$\because$ AC = AB + BC

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

$\Rightarrow |\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$

Combining (1) and (2), we get

$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$

(ii) $|\vec{a}| = |\vec{a}-\vec{b}+\vec{b}| = |(\vec{a}-\vec{b})+\vec{b}|$...............(1)

But $|(\vec{a}-\vec{b})+\vec{b}| \le |\vec{a}-\vec{b}| + |\vec{b}|$................(2)

From (1) and (2), we get

$|\vec{a}| \le |\vec{a}-\vec{b}| + |\vec{b}|$

$|\vec{a}| - |\vec{b}| \le |\vec{a}-\vec{b}|$

(iii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \le |\vec{a}| + |-\vec{b}|$

But $|-\vec{b}| = |\vec{b}|$

$\therefore |\vec{a}-\vec{b}| \le |\vec{a}| + |\vec{b}|$.

Example – 2: **Prove by vector method that the lines segment joining the middle points of any two sides of a triangle is parallel to the third side and equal to half of it.**

Solution : Let ABC be a triangle in which D and E are the mid-points of AB and AC respectively. ***(fig.16)***

$$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$$

$$= \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) = \frac{1}{2}\overrightarrow{BC}$$

$\therefore$ DE || BC

Also, DE $= |\overrightarrow{DE}| = \left|\frac{1}{2}\overrightarrow{BC}\right| = \left|\frac{1}{2}\right| |\overrightarrow{BC}| = \frac{1}{2}$ BC

Hence DE || BC and DE $= \frac{1}{2}$ BC.

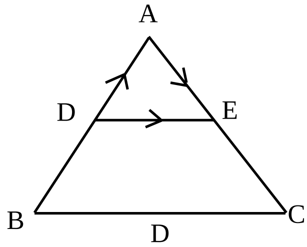


Fig. – 16

Components of a Vector in Two Dimensions

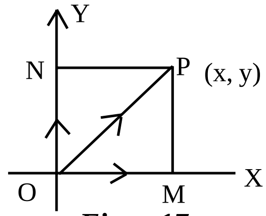


Fig. – 17

Let XOY be the co-ordinate plane let P(x, y) be a point in this plane. Join P. Draw PM $\perp$ OX, and PN $\perp$ OY.***(fig.17)***

Let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY.

Then $\vec{OM} = x\hat{i}$ and $\vec{ON} = y\hat{j}$.

$\vec{OM}$ and $\vec{ON}$ are called the vector components of $\vec{OP}$ along x-axis and y - axis respectively.

Thus the component of $\vec{OP}$ along x- axis is a vector x, whose magnitude is | x | and whose direction is along OX and OX′ according as x is positive or negative.

And, the component of $\vec{OP}$ along y - axis is a vector y, whose magnitude is | y | and whose direction is along OY or OY′ according as y is positive or negative.

$$\vec{OP} = \vec{OM} + \vec{MP} = \vec{OM} + \vec{ON} = x\hat{i} + y\hat{j}$$

Thus the position vector of the point P(x, y) is $x\hat{i} + y\hat{j}$

$OP^2 = OM^2 + MP^2 = x^2 + y^2$

$\Rightarrow OP = \sqrt{x^2 + y^2}$

$\therefore |\vec{OP}| = \sqrt{x^2 + y^2}$

Components of a vector along the co-ordinate axes.

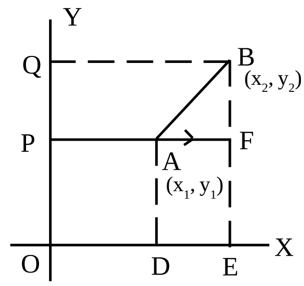


Fig. –18

Let $A(x_1,y_1)$ and $B(x_2, y_2)$ be any two points in XOY plane.

Draw AD $\perp$ OX, ***(fig.18)***

BE $\perp$ OX AF $\perp$ BE, AP $\perp$ OY and BQ $\perp$ OY

Clearly $AF = (x_2 - x_1)$

and $PQ = FB = (y_2 - y_1)$

Let $\hat{i}$ and $\hat{j}$ be unit vectors along x-axis and y-axis respectively.

Then $\overrightarrow{AF} = (x_2 - x_1)\,\hat{i}$

and $\overrightarrow{PQ} = \overrightarrow{FB} = (y_2 - y_1)\,\hat{j}$

Clearly $\overrightarrow{AB} = \overrightarrow{AF} + \overrightarrow{FB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$

Then component of $\vec{AB}$ along x - axis = $(x_2 - x_1)\,\hat{i}$

And component of $\vec{AB}$ along y - axis = $(y_2 - y_1)\,\hat{j}$

Also $|\overrightarrow{AB}| = AB = \sqrt{AF^2 + FB^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Components of Vector in three dimensions :

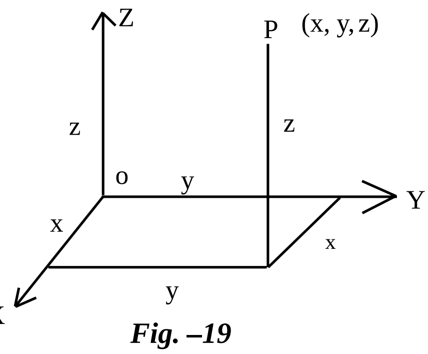


Fig. –19

Let OX, OY and OZ be three mutually perpendicular lines, taken as co-ordinate axis. Then the planes XOY, YOZ and ZOX are respectively known as XY plane, YZ plane and ZX plane. ***(fig.19)***

Let P be any point in space. Then the distances of P from YZ- plane. ZX – plane and XY - plane are respectively called x-cordinate, y-cordinate and z-cordinate of P and we write P as P(x, y, z)

Position Vector of Point in space :

Let P(x, y, z) be a point in space with reference to three co-ordinate axes. OX, OY and OZ. Though P draw planes parallel to yz-plane zx-plane and xy-plane meeting the axes OX, OY and OZ at A, B and C respectively.

The OA = x, OB = y and OC = z

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vector along OX, OY and OZ respectively. ***(fig. 20)***

Then $\overrightarrow{OA} = x\,\hat{i}$, $\overrightarrow{OB} = y\,\hat{j}$, $\overrightarrow{OC} = z\,\hat{k}$

Now $\overrightarrow{OP} = \overrightarrow{QP} + \overrightarrow{QP} = (\overrightarrow{OA} + \overrightarrow{AQ}) + \overrightarrow{QP}$

$(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$ $\left[\because \overrightarrow{AQ} = \overrightarrow{OB} \text{ and } \overrightarrow{QP} = \overrightarrow{OC}\right]$

$= x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

Thus, the position vector of a point

P (x, y, z) is the vector $(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$

Now $OP^2 = OQ^2 + QP^2 = (OA^2 + AQ^2) + QP^2$

$= (OA^2 + OB^2 + OC^2) = x^2 + y^2 + z^2$

$OP = \sqrt{x^2 + y^2 + z^2}$

$|\overrightarrow{OP}| = OP = \sqrt{x^2 + y^2 + z^2}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$,

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

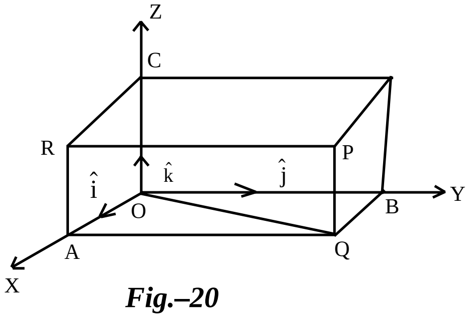


Fig.–20

Components of Vector : If $\overrightarrow{OP}$ is the position vector of a point P(x, y, z) in space, then

$\overrightarrow{OP} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

The vectors $x\,\hat{i}$, $y\,\hat{j}$, $z\,\hat{k}$ are called the components of $\overrightarrow{OP}$ along x - axis y - axis and z-axis respectively.

ASSIGNMENTS

1. Show that the there points A(2, –1, 3), B (4, 3, 1) and C (3, 1, 2) are co-llinear.

2. Prove by vector method that the medians of a triangle are concurrent.

3. Find a unit vector in the direction of $(\vec{a} + \vec{b})$ where $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ & $\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$.

Scalar or Dot Product

Definition :

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ with magnitude a and b respectively, denoted by $\vec{a}\cdot\vec{b}$, is defined as the scalar ab cos θ , where θ is the angle between of $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$.

Thus $\vec{a}\,.\,\vec{b} = ab\cos\theta$.

Geometrical Meaning of Scalar Product

As we see in above figure that ***(fig.21)***

$|OM| = |OB| \cos\theta = |\vec{b}| \cos\theta$ = projection of $\vec{b}$ on $\vec{a}$

$\therefore$ a.b= $|\vec{a}|\,(|\vec{b}|\cos\theta)$

= modulus of $\vec{a}$ × projection of $\vec{b}$ on $\vec{a}$ which gives,

projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

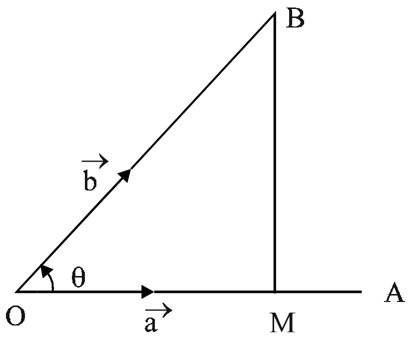


Fig. – 21

Similarly , if we drop a perpendicular from A on OB such that N is the foot of the perpendicular, then ***(fig.22)***

ON = projection of $\vec{a}$ on $\vec{b}$ and ON = OA cos θ

$= |\vec{a}| \cos\theta$ = Now $\vec{a} \cdot \vec{b} = |\vec{b}|(|\vec{a}|\cos\theta)$

= magnitude of $\vec{b}$ × projection of $\vec{a}$ on $\vec{b}$ which gives that

projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

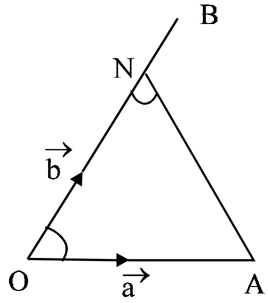


Fig. – 22

Thus we can conclude that

(i) The dot product of two vectors is equal to the magnitude of one vector multiplied by the projection of the other on it.

(ii) The (scalar) projection of one vector on another.

$= \dfrac{\text{Dot product of vectors}}{\text{Magnitude of the vector}}$ on which the projection is taken.

3. Commutative and distributive Properties of Scalar Product :

1. Scalar product of two vectors obeys commutative law i.e.,

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

2. Scalar product obeys distributive law i.e.

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

Other properties of scalar product : Apart from commutative and distributive properties. Scalar product has some ther properties as follow :

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2 \Rightarrow \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

2. $\vec{a}\,.\,\vec{b} = 0 \Rightarrow \vec{a}$ is $\perp$ to $\vec{b}$.

   Hence $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ and $\hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{i}\cdot\hat{k} = 0$

3. Scalar product in terms of components :

   If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

   then $\vec{a}\,.\,\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

4. Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab} = \hat{a}\cdot\hat{b}$

   In terms of components $\cos\theta = \dfrac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$

5. Projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ $\vec{a}\,,\hat{b}$ and projection of $\vec{b}$ on $\vec{a}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \hat{a}.\vec{b}$

6. $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\,.\,\vec{b}$ or $(\vec{a}+\vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a}\,.\,\vec{b}$

7. $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\,.\,\vec{b}$ or $(\vec{a}-\vec{b})^2 = \vec{a}^2 + \vec{b}^2 - 2\vec{a}\,.\,\vec{b}$

8. $(\vec{a}+\vec{b})\,.\,(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$ or $(\vec{a}+\vec{b})\,.\,(\vec{a}-\vec{b}) = \vec{a}^2 - \vec{b}^2$

Components of a Vector $\vec{r}$ along and perpendicular to a given Vector $\vec{a}$ in the Plane of $\vec{a}$ and $\vec{r}$.

The resolved part of $\vec{r}$ in the direction of $\vec{a}$ = $\left(\dfrac{\vec{a}\cdot\vec{r}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$

The resolved part of r is perpendicular to a is $\left(\dfrac{\vec{r}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$

Example –1: Find the scalar and vector projections of $\hat{i}-\hat{j}-\hat{k}$ on $\hat{i}+\hat{j}+3\hat{k}$

Solution : Given $\vec{a}=\hat{i}-\hat{j}-\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-3\hat{k}$

Scalar projection of $\vec{a}$ on $\vec{b}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$=\frac{(\hat{i}-\hat{j}-\hat{k})(3\hat{i}+\hat{j}+3\hat{k})}{\sqrt{3^2+1^2+3^2}}=\frac{3-1-3}{\sqrt{19}}=\frac{-1}{\sqrt{19}}$$

Vector Projection of $\vec{a}$ on $\vec{b}=\left(\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\cdot\dfrac{\vec{b}}{|\vec{b}|}$

$$=\left(\frac{-1}{\sqrt{19}}\right)\cdot\left(\frac{3\hat{i}+j+3\hat{k}}{\sqrt{3^2+1^2+3^2}}\right)$$

$$=\frac{-1}{\sqrt{19}}\left(\frac{3\hat{i}+\hat{j}+3\hat{k}}{\sqrt{19}}\right)=-\left(\frac{3}{19}\hat{i}+\frac{\hat{j}}{19}+\frac{3\hat{k}}{19}\right)$$

ASSIGNMENTS

1. $\vec{a},\vec{b},\vec{c}$ are there rutually perpendicular vectors of the same magnitude prove that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined in the vectors $\vec{a},\vec{b}$ & $\vec{c}$.

2. Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a}=\hat{i}-\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+3\hat{k}$

3. Find the angle between the vector $\vec{a}=-\hat{i}+\hat{j}-2\hat{k}$ & $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$

Vector Product or Cross Product

The vector product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\times\vec{b}$ is defined as the vector $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|$ $\sin\theta$. $\hat{n}$ where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle from $\vec{a}$ and $\vec{b}$ such that $\vec{a}$ and $\vec{b}$ and $\hat{n}$ are the right handed system.

Angle between two vectors :

Let θ be the angle between $\vec{a}$ and $\vec{b}$. The $\vec{a}\times\vec{b}$ = (ab $\sin\theta$) $\hat{n}$, where $|\vec{a}| = a$ and $\vec{b} = b$

$\therefore\ |\vec{a} \times \vec{b}| = (ab \sin\theta)\,|\hat{n}| = ab \sin\theta$

$[\because\ |\hat{n}| = 1]$

$ab \sin\theta = |\vec{a} \times \vec{b}|$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{ab} = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\theta = \sin^{-1}\left\{\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right\}$$

Unit vector perpendicular to two vectors :

Clearly $(\vec{a} \times \vec{b})$ is a vector, perpendicular to each one of the vector $\vec{a}$ and $\vec{b}$, so a unit vector $\hat{n}$ perpendicular to each one of the vector $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Properties of vector product :

(i) Vector product is not commutative

i.e. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(ii) For any vecrors $\vec{a}$ and $\vec{b}$ i.e. $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

(iii) For any scalar m prove that $(m\vec{a}) \times \vec{b} \Rightarrow m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$

(iv) For any vectors $\vec{a}, \vec{b}, \vec{c}$ present $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

(v) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ $\vec{a} \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})$

(vi) The vector product of two parallel or collinear vectors is zero.

(vii) For any vector $\vec{a}$ is $\vec{a} \times \vec{a} = \vec{0}$

(viii) If $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ are the parallel or collinear.

(ix) If the vectors $\vec{a}$ and $\vec{b}$ are parallel (or collinear) then θ = 0 or 180°, sinθ = 0

Vector product of orthonomal Triad of unit vectors :

Vector products of unit vectors $\hat{i}, \hat{j}, \hat{k}$ from a right-handed system of mutually perpendicular vectors. ***(fig.23)***

$\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$

$\hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times j$

$\hat{k} \times \hat{i} = j = -\hat{i} \times \hat{k}$

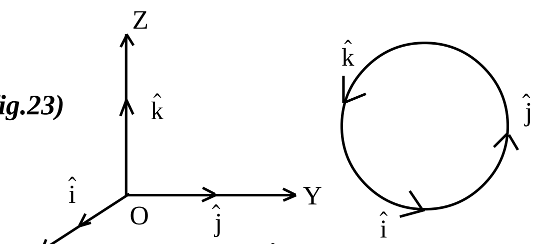


Fig.–23

Geometrical Interpretation of Vector Product or Cross Product

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Then $\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\ \hat{n}$

$= |\vec{a}|(|\vec{b}| \sin\theta)\ \hat{n} = |\vec{a}|\,|BM|\ \hat{n}$

Now $|\vec{a} \times \vec{b}| = |\vec{a}|\,|BM|$

= Area of the parallelogram with sides $\vec{a}$ and $\vec{b}$. ***(fig 24)***

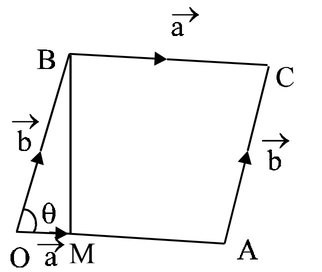


Fig. –24

Therefore, $\vec{a} \times \vec{b}$ is a vector whose magnitude is equal to area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.

From this it can be concluded that Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

Example – 1: Find the area of parallelogram whose adjacent sides are determined by the vectors.

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$$

Solution : We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = (8\hat{i} - 10\hat{j} + 4\hat{k})$

$\therefore$ Required area $= |\vec{a} \times \vec{b}|$

$= \sqrt{8^2 + (-10)^2 + 4^2} = \sqrt{180} = 6\sqrt{5}$ sq. units

Example – 2: Find the area of a parallelogram whose diagonals are determined by the vectors.

$$\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k} \text{ and } \vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$$

Solution : We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (-2\hat{i} - 14\hat{j} - 10\hat{k})$

$\therefore$ Required area $= \frac{1}{2}|\vec{a} \times \vec{b}|$

$= \frac{1}{2}\sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \frac{1}{2}\sqrt{300} = 5\sqrt{3}$ sq. units.

ASSIGNMENTS

1. Find the area of the triangle whose adjacant sides are $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ & $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$

2. Find a unit vector perpendicular to both the vector $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ & $\vec{b} = 3\hat{i} - \hat{j} + 3\hat{k}$

3. Find the angle between the vectors $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ & $\vec{b} = \hat{i} + 3\hat{j} + 2\hat{k}$

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