Bending Stress

When a beam is loaded with external loads, all the sections of the beam will experience bending moments and shear forces. The shear forces and bending moments at various sections of the beam can be evaluated as discussed in the earlier chapter. In this chapter, the bending and bending stress distribution across a section will be dealt with.

Some practical applications of bending stress shall also be dealt with. These are

1. Moment carrying capacity of a section
2. Evaluation of extreme normal stresses due to bending
3. Design of beam for bending
4. Evaluation of load bearing capacity of the beam

The major stresses induced due to bending are normal stresses of tension and compression. But the state of stress within the beam includes *shear stresses due to the shear force* in addition to the major *normal stresses due to bending* although the former are generally of *smaller order* when compared to the latter.

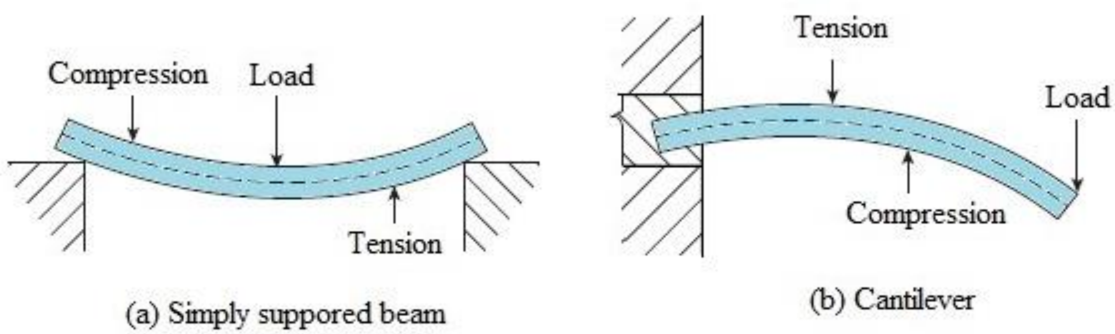


Fig. 1

Simple Bending or Pure Bending

A beam or a part of it is said to be in a state of pure bending when it bends under the action of uniform/constant bending moment, without any shear force.

Alternatively, a portion of a beam is said to be in a state of simple bending or pure bending when the shear force over that portion is zero. In that case there is no chance of shear stress in the beam. But, the stress that will propagate in the beam as a result will be known as normal stress.

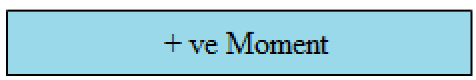
However, in practice, when a beam is subjected to transverse loads, the bending moment at a section is accompanied by shear force. But, it is generally observed that the shear force is zero where the bending moment is maximum. Therefore, the condition of pure bending or simple bending is deemed to be satisfied at that section.

Examples of pure bending are –

1. Bending of simple supported beam due to end coupling (Uniform pure bending)
2. Bending of cantilever beam with end moment (Uniform pure bending)
3. Bending of the portion between two equal point loads in a simple supported beam with two-point loading (Non-uniform pure bending)

The *four point bending* of the simply supported beam

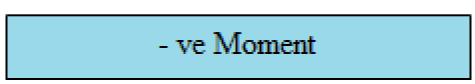
(a) Simple supported beam with end coupling



(b) Bending moment diagram

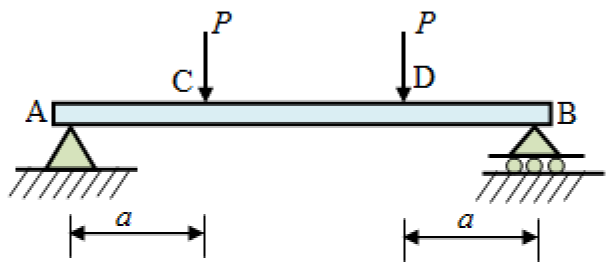
Fig. 2

(a) Cantilever subjected to moment at its end

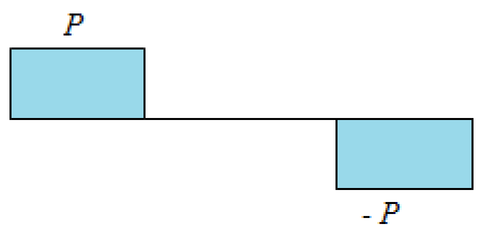


(b) Bending moment diagram

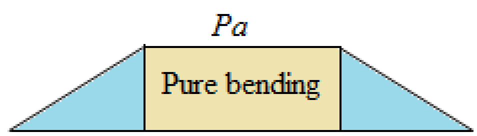
Fig. 3



(a) Simply supported beam with two-point loading



(b) Shear force diagram



(c) Bending moment diagram

Fig. 4

Theory of Simple Bending

The theory which deals with the determination of stresses at a section of a beam due to pure bending is called ***theory of simple bending***. In this chapter, bending of *straight homogeneous* beams of *uniform cross sectional area* with vertical *axis of symmetry* shall be considered. The application of this theory can be extended to beams with two or more different materials as well as curved beams.

Several cross-sections of beams satisfying the above conditions are shown in the Fig. 5.

A beam of rectangular cross-section with typical loading condition is shown in the Fig. 6. Also shown in the Fig. 7 is the three-dimensional beam with longitudinal vertical plane of symmetry, with the cross-section symmetric about this plane. It is assumed that the loading and supports are also symmetric about this plane. With these conditions, the beam has no tendency to twist and will undergo bending only.

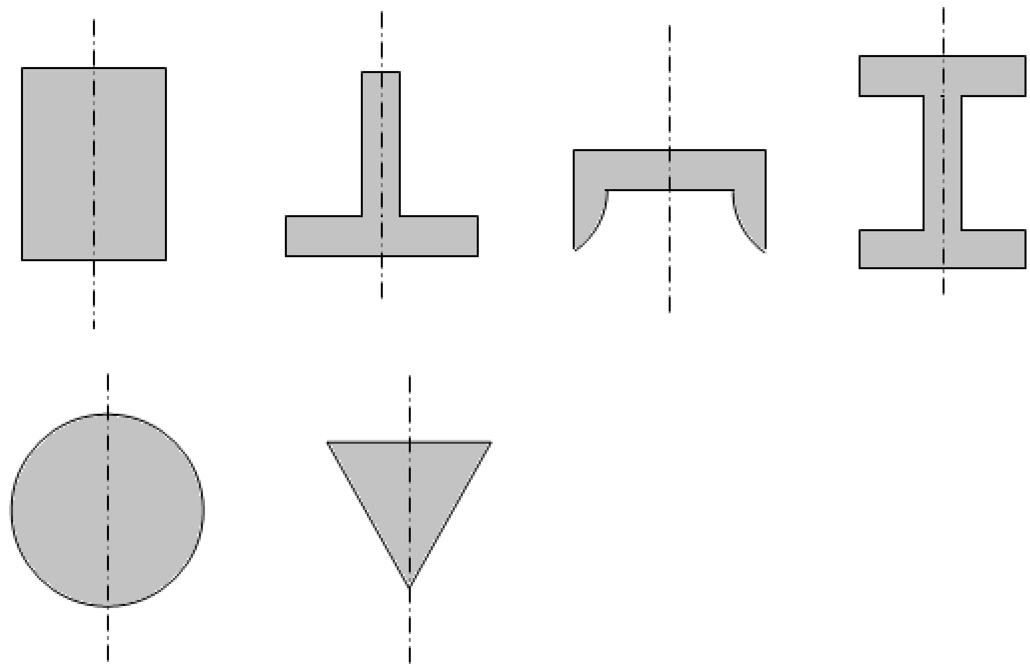


Fig. 5 Beam cross-sections with vertical axis of symmetry

A beam subjected to sagging moment is shown in the Fig. 8. The beam is imagined to be consisting of a number of longitudinal fibres; one such fibre is is shown in colour. It is obvious that the fibres near the upper side of the beam are compressed; hence an element in the upper part is under compression. The fibres at the bottom side of the beam get stretched and, hence, the elements on the lower side are subjected to tension. Somewhere in between, there will be a plane where the fibres are subjected to neither tension nor compression. Such a plane is termed as ***neutral surface*** *or* ***neutral plane***.

In the conventional coordinate system attached to the beam in Fig. 8, *x* axis is the longitudinal axis of the beam, the *y* axis is in the transverse direction and the longitudinal plane of symmetry is in the *x*- *y* plane, also called the ***plane of bending***.

Neutral Surface

The longitudinal surface of a beam under bending which experiences neither tension nor compression is known as neutral surface. There is only one neutral surface in a beam.

Neutral Axis

The line of intersection of transverse section of beam with the neutral surface is known as neutral axis. In other words, the line of intersection of the longitudinal plane of symmetry and the neutral surface is known as neutral axis. Neutral axis experiences no extension or contraction.

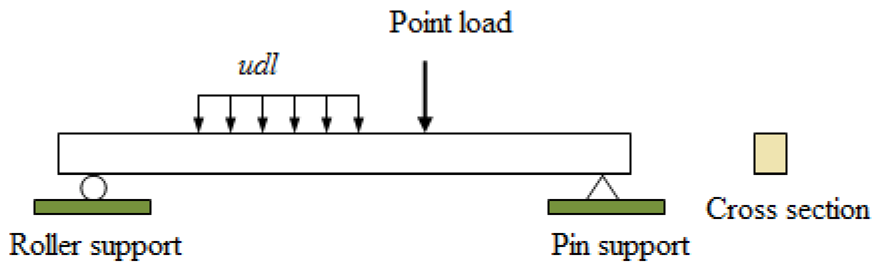


Fig.6 Loaded beam

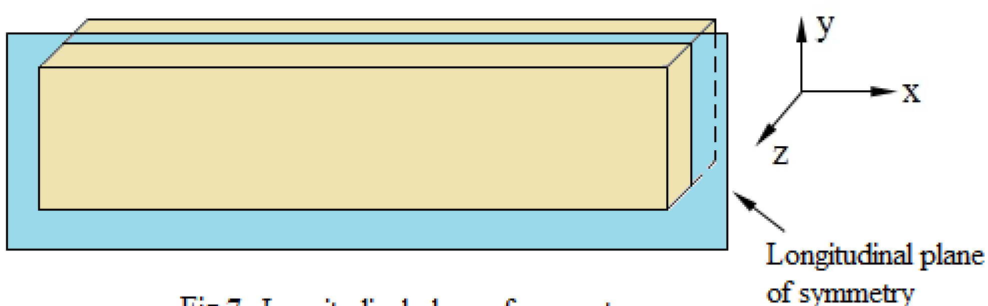


Fig.7 Longitudinal plane of symmetry

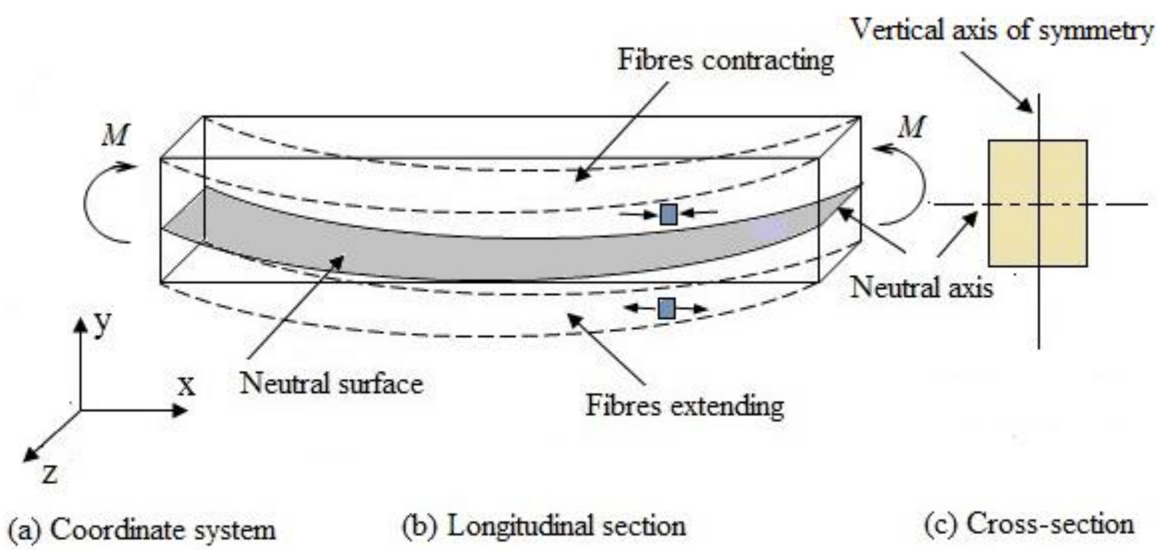


Fig. 8 Pure moment (sagging)

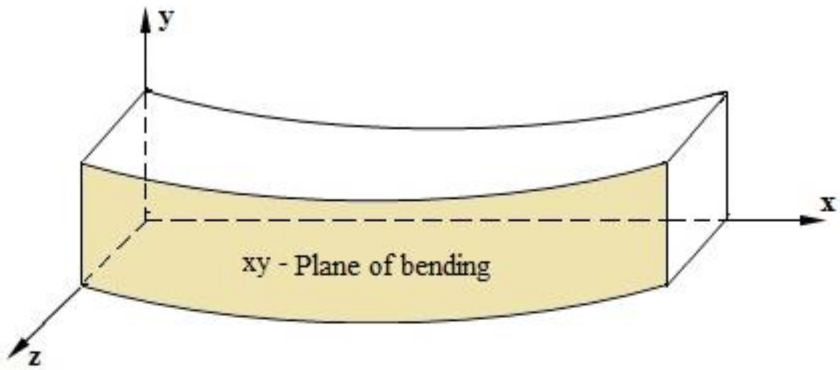


Fig. 9 Plane of bending

Axis of beam

The intersection of the longitudinal plane of symmetry and the neutral surface is called the **axis of the beam.** In other words, the line through the centroid of all the cross-sections of the beam is known as axis of the beam.

Assumptions for theory of pure bending:

The assumptions made in the theory of simple bending are as follows:

1. The material of the beam is perfectly homogeneous (i.e. of the same kind throughout) and isotropic (i.e. of same elastic properties in all directions).
2. The material is stressed within elastic limit and obeys Hooke's law.
3. The value of modulus of elasticity for the material is same in tension and compression.
4. The beam is subjected to pure bending and therefore bends in the form of an arc of a circle.
5. The transverse sections, which are plane and normal to the longitudinal axis before bending, remain plane and normal to the longitudinal axis of the beam after bending.
6. The radius of curvature of the bent axis of the beam is large compared to the dimensions of the section of beam.
7. Each layer of the beam is free to expand or contract independently.
8. The cross-sectional area is symmetric about an axis perpendicular to the neutral axis.

Explanation of the assumptions

According to assumption No. 5, plane section *ABCD* before bending as shown in Fig. 10 remains plane after bending as shown by *A'B'C'D'*. This assumption, also known as Bernoulli's assumption, is perfectly valid for beams with pure bending. If there is any shear along with the bending, the shear deformation distorts the plane and *A'B'* will not remain plane. However, for beams with smaller depth (d<1/10th span) shear deformation is small and this assumption is not much affected. In case of deep beams, with shear forces, this assumption fails.

Assumption No. 6, the radius of curvature is large compared to depth is valid if deflections are less than 1/10th to 1/5th of depth of beam. Therefore, the theory derived with this assumption may be called ***small deflection theory***.

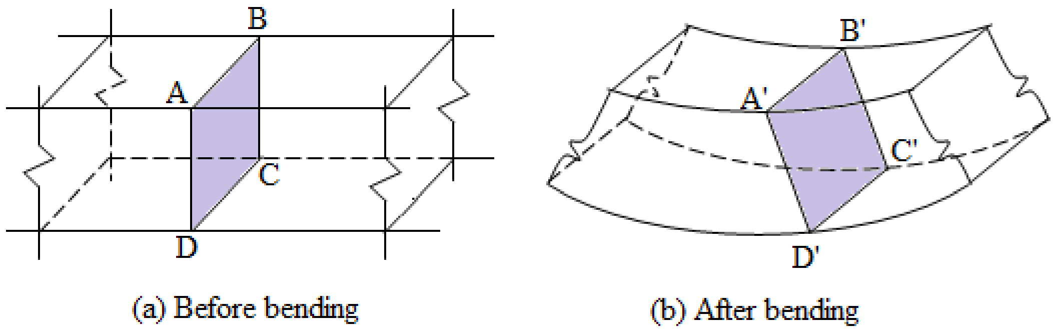


Fig. 10

Relationship between Bending Stress and Radius of Curvature

Consider a part of beam *ABCD* of length *dx* subjected to pure bending of bending moment *M* as shown in the Fig. 11. As the beam is subjected to pure bending, it bends into a circular arc.

The topmost layer *AB* is contracted to *A'B'*. The layer *PQ* below it is compressed to a lesser degree than it. The bottom most layer *CD* is elongated to *C'D'*. All other layers are subjected to different degrees of elongation or contraction degrees depending upon their position. However, one layer *MN* has not suffered any change in its length. This layer is called the ***neutral layer*** or ***neutral surface***.

Let $d\theta$ be the angle formed by the planes *A'C'* and *B'D'* and *R* be the radius of the neutral layer. Consider a fibre *PQ* at a distance of y from the neutral layer.

Original length of the fibre, $PQ = dx = Rd\theta$

After deformation, the length of the fibre is compressed to P'Q'.

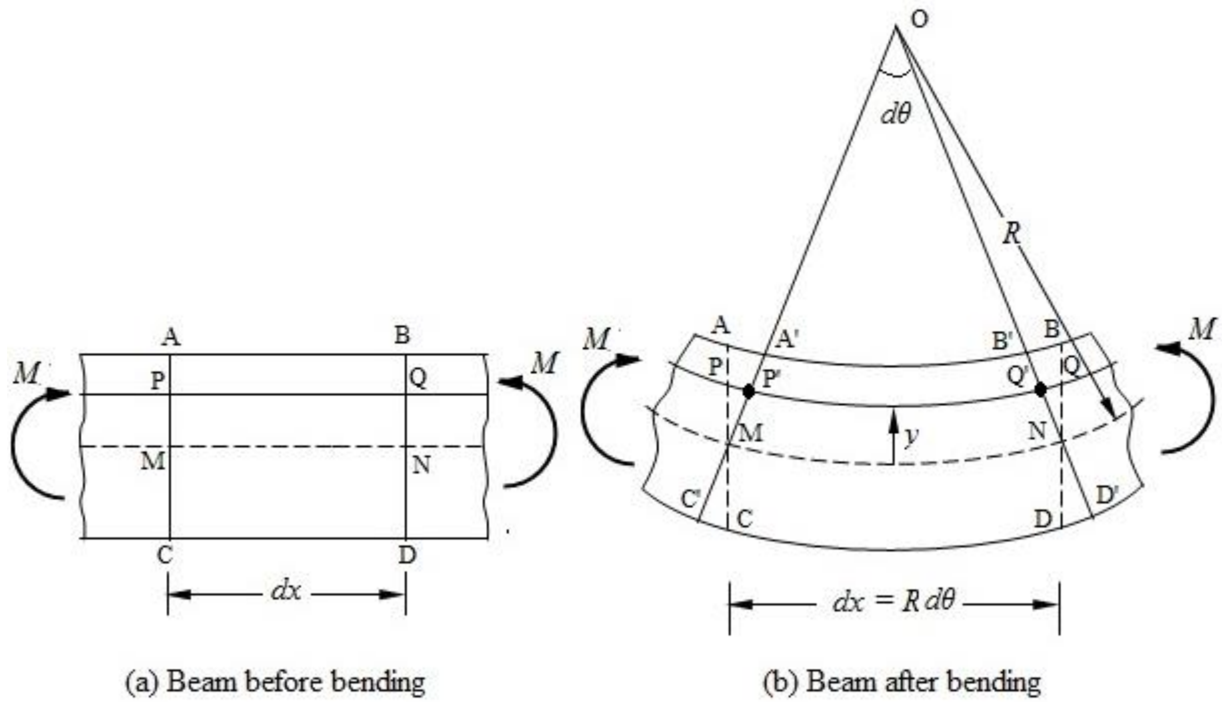


Fig. 11

Decrease in length of the fibre $PQ = PQ - P'Q'$

$$= R\,d\theta - (R - y)\,d\theta$$

$$= y\,d\theta$$

Let the projection of *C' A'* and *D' B'* meet at *O*.

Strain in the fibre *PQ*, $\varepsilon = \dfrac{\text{Decrease in length}}{\text{Original length}}$

$$\therefore\ \varepsilon = \frac{y\,d\theta}{R\,d\theta} = \frac{y}{R}$$

Let σ be the stress in the fibre *PQ*.

Then, $\varepsilon = \dfrac{\sigma}{E}$, where E is the Modulus of elasticity of the material.

$$\therefore\ \varepsilon = \frac{\sigma}{E} = \frac{y}{R}$$

$$\therefore\ \sigma = \frac{E}{R} \times y$$

Hence, the stress intensity in any fibre is proportional to the distance of the fibre from the neutral layer.

Position of Neutral Axis

Consider a beam of arbitrary cross-section as shown in the Fig. 12. Consider an elemental are δa at a distance y from the neutral axis. Let the bending stress on the element be σ.

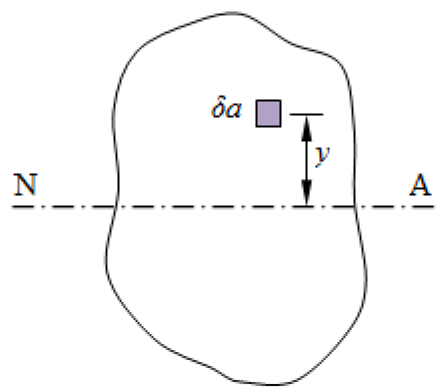


Fig. 12

Force on the elemental area $= \sigma\,\delta a$

Force over the entire cross-section of the beam $= \sum \sigma\,\delta a$

We also know, $\sigma = \dfrac{E}{R} \times y$

Substituting the value of σ, we get

Force over the entire cross-section of the beam $= \sum \dfrac{E}{R} y\delta a = \dfrac{E}{R} \sum y\delta a$

Since there is no axial force on the beam, from equilibrium consideration, the above axial force should be zero.

Hence, $$\frac{E}{R}\sum y\delta a = 0$$

Since, $\dfrac{E}{R}$ is constant for a given section, we have $\sum y\delta a = 0$

We know, $$A\bar{y} = \sum y\delta a$$

Where, A is the area of cross-section of the beam.

So, $$A\bar{y} = 0$$

$$\text{or} \qquad \bar{y} = 0$$

$\bar{y}$ is the distance of the centroid from the neutral axis. Hence, the neutral axis of the section coincides with the centroid of the section. Thus, to locate the neutral axis of a section, the centroid of the section should be determined. The line passing through the centroid, parallel to the plane of bending is the neutral axis of the beam section.

Relationship between Moment and Radius of Curvature

Consider an elemental area δa from the neutral axis of a beam section as shown in the Fig. 13.

The stress on the elemental area, $\boldsymbol{\sigma} = \frac{E}{R} y$

Force on the elemental are $\boldsymbol{\sigma}\, \delta a = \frac{E}{R} y\, \delta a$

Moment of resistance offered by this elemental area about the neutral axis

$$= \left(\frac{E}{R} y\, \delta a\right) y = \frac{E}{R} y^2 \delta a$$

Total moment of resistance, M offered by the cross-sectional area of beam,

$$M = \sum \frac{E}{R} y^2 \delta a$$

$$M = \frac{E}{R} \sum y^2 \delta a$$

But, $\sum y^2 \delta a$ is the moment of inertia I of the beam section about the neutral axis.

$$\therefore \quad M = \frac{E}{R} I$$

$$\frac{M}{I} = \frac{E}{R}$$

We have earlier seen that, $\frac{\boldsymbol{\sigma}}{y} = \frac{E}{R}$

Combining the two equations, we get

$\frac{M}{I} = \frac{\boldsymbol{\sigma}}{y} = \frac{E}{R}$, which is known as the *bending equation*.

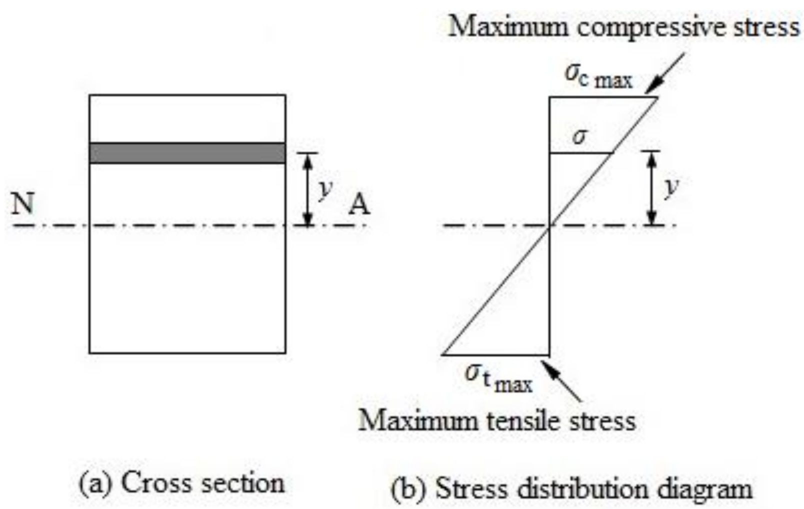


Fig. 13

Where, M = bending moment at a section,

I = moment of inertia of the beam section,

σ = stress at any layer of the beam,

y = distance of the layer from the neutral axis,

E = Young's Modulus and

R = radius of curvature.

M and I are constants for a particular beam section. Hence, σ varies proportionally to the distance y. So, maximum stress occurs at extreme fibres. The stress distribution will be triangular as shown in the Fig. 13.

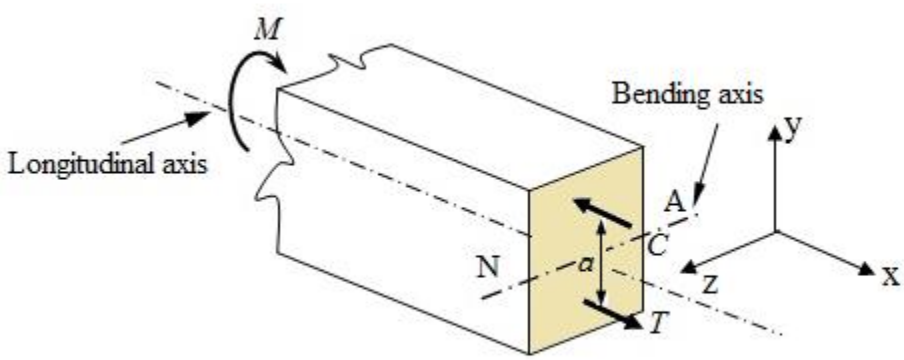


Fig. 14

The formula for flexural stress derived as above applies only to cases where the material behaves elastically. The important concepts used in deriving the flexural formula may be summed up as follows.

1. Strains in different layers of beam vary linearly with their distances from the neutral axis.
2. Properties of materials are used to relate strain and stress.
3. Equilibrium conditions are used to locate the neutral axis and to determine the internal stresses.

The internal bending moment developed by the induced flexural stresses due to bending at a section is known as ***moment of resistance*** of the section. For equilibrium of the section, the moment of resistance of a section should be equal to or greater than the applied external moment.

Flexural rigidity:

From equation of flexure, we have

$$\frac{M}{I} = \frac{E}{R}$$

$$EI = MR$$

EI is known as flexural rigidity. ***Flexural rigidity*** is the measure of flexural strength of a beam section. Higher is the flexural rigidity better is the flexural strength. It depends upon the material as well as the geometric property of the section. Elastic modulus, E reflects the material character and moment of inertia, I reflects the geometric characteristic

Economical section

In a beam of rectangular or circular section, the fibres near neutral axis are under-stressed compared to those at the top and bottom. As a result, a large portion of the beam cross-section remains under-stressed and under utilized for resisting flexure or bending.

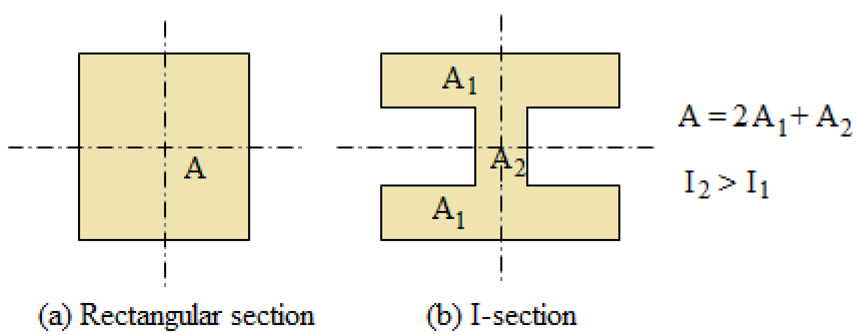


Fig. 15

The expression $M = \frac{\sigma}{y} I$ indicates that moment of resistance of a section can be greatly increased by increasing the moment of inertia by rearranging or redistributing the area while keeping the cross-sectional area and the depth of the beam unchanged. This can be achieved by changing the geometry of the section so as to spread the area farther from the neutral axis.

In order to increase the moment of resistance to bending of a beam section, it is advisable to use sections which have large area away from the neutral axis. Hence, I-section and T-sections are preferable to rectangular section.

Sections of different geometry, (i) rectangular section and (ii) I-section of equal cross-sectional area and same depth are shown in the Fig. 15.

Moment carrying capacity of a section:

From equation of flexure, we have

$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M}{I} y$$

It is obvious that bending stress is maximum on the extreme fibre at the top and bottom of the beam where y is maximum. In design of beam, the extreme fibre stress should not be allowed to exceed the allowable or permissible stress of the material. If σ_{allow} is the allowable stress for bending, then for safe design

$$\sigma_{\max} \leq \sigma_{allow}$$

$$\frac{M}{I} y_{\max} \leq \sigma_{allow}$$

If M is taken as the maximum moment carrying capacity of the section,

$$\frac{M}{I} y_{\max} \leq \sigma_{allow}$$

$$M \le \frac{I}{y_{\max}} \sigma_{allow}$$

The moment of inertia I and the extreme fibre distance $y_{\max}$ are the geometrical properties of the section. The ration of the moment of inertia and the extreme fibre distance $\left(I/y_{\max}\right)$ for a given cross-section of beam is constant and is known as ***section modulus*** (Z). Thus the moment carrying capacity of a beam is given by

$$M = \sigma_{allow} Z$$

If σ_{allow} in tension and compression are same, doubly symmetric section is selected. Doubly symmetric section means a section which is symmetric about the vertical as well as neutral axis. If σ_{allow} in tension and compression are different, un-symmetric cross-section is selected such that the distance to the extreme fibers are nearly the same ratio as the respective allowable stresses. In the latter case, the moment carrying capacity in tension and compression are found separately and the smaller one is taken as the moment carrying capacity of the section.

Section Modulus of Sections of Standard Geometry

1. Rectangular section

Let us consider a rectangular section of width b and depth d as shown in the Fig. The neutral axis coincides with the centroidal axis of the beam.

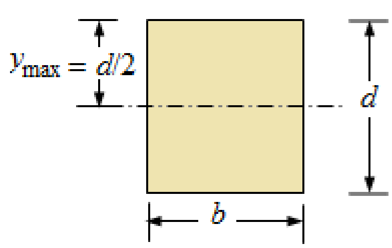


Fig. 16

Moment of inertia about the neutral axis, $I = \dfrac{bd^3}{12}$

Distance of outermost fibre from the neutral axis, $y_{\max} = \dfrac{d}{2}$

Section modulus, $$Z = \frac{I}{y_{\max}} = \frac{bd^3}{12} \times \frac{2}{d}$$

$$= \frac{1}{6} bd^2$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \frac{1}{6} \sigma bd^2$

2. Hollow Rectangular section

Let us consider a hollow rectangular section of size $B \times D$ with a symmetrical opening $b \times d$ as shown in the Fig. 17.

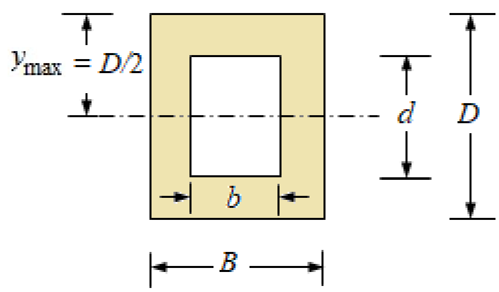


Fig. 17

Moment of inertia about the neutral axis, $I = \frac{BD^3}{12} - \frac{bd^3}{12}$

Distance of outermost fibre from the neutral axis, $y_{max} = \frac{D}{2}$

Section modulus, $$Z = \frac{I}{y_{max}} = \frac{BD^3 - bd^3}{12} \times \frac{2}{D}$$

$$= \frac{1}{6} \frac{\left(BD^3 - bd^3\right)}{D}$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \frac{1}{6} \sigma \frac{(BD^3 - bd^3)}{D}$

3. Circular section

Let us consider a circular section of diameter d as shown in the Fig. 18.

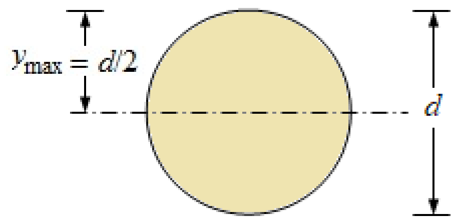


Fig. 18

Moment of inertia about the neutral axis, $I = \dfrac{\pi d^4}{64}$

Distance of outermost fibre from the neutral axis, $y_{max} = \dfrac{d}{2}$

Section modulus,

$$Z = \frac{I}{y_{max}} = \frac{\pi d^4}{64} \times \frac{2}{d}$$

$$= \frac{\pi d^3}{32}$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \sigma \dfrac{\pi d^3}{32}$

4. Hollow Circular section

Let us consider a hollow circular section of external and internal diameter D and d respectively as shown in the Fig. 19.

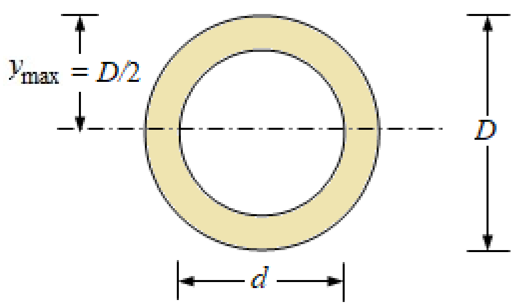


Fig. 19

Moment of inertia about the neutral axis, $I = \dfrac{\pi}{64}\left(D^4 - d^4\right)$

Distance of outermost fibre from the neutral axis, $y_{\max} = \dfrac{D}{2}$

Section modulus, $$Z = \frac{I}{y_{\max}} = \frac{\pi}{64}\left(D^4 - d^4\right)\times\frac{2}{D}$$

$$= \frac{\pi}{32D}\left(D^4 - d^4\right)$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \dfrac{\sigma\pi}{32D}\left(D^4 - d^4\right)$

Design of beam for bending

Design of beam involves the determination of the size (cross-section) of the beam for given loading condition. The maximum bending moment of the beam is determined from the loading condition. Given the bending moment and permissible bending stress of the material of the beam, the section modulus of the beam is determined from the expression of bending stress. Once the section modulus is known, width and depth can easily determined assuming the depth to width ratio.

Beam of uniform strength

In practice, a beam of uniform cross section is designed for moment of resistance same as the maximum bending moment that the beam is supposed to carry. Hence, the material in all sections except the section of maximum bending moment remains under-stressed and underutilized. Although practical, such a beam is uneconomical. Ideally, a beam of varying cross-section should be designed so that all sections attain the maximum permissible stress simultaneously. A beam in which permissible stress at all sections is reached simultaneously under a given loading, is called a ***beam of uniform strength***.

A beam of uniform strength can be obtained in different ways

a) By varying the width of beam and keeping the depth constant
b) By varying the depth of beam and keeping the width constant
c) By varying both width and depth

By varying the width of beam and keeping the depth constant

Derive the formula for cross section of a rectangular beam of uniform strength for a cantilever beam of length L carrying concentrated load at free end by keeping the depth constant.

Consider a cantilever beam of length L and uniform depth d carrying a concentrated load W at its free end as shown in the Fig. 20. Let the width varies from a minimum at its free end to a maximum of b near the fixed end.

It is obvious that the bending moment varies from minimum zero at the free end to maximum at WL at the fixed support.

Bending moment at any section at a distance of x from the free end,

$$M = Wx$$

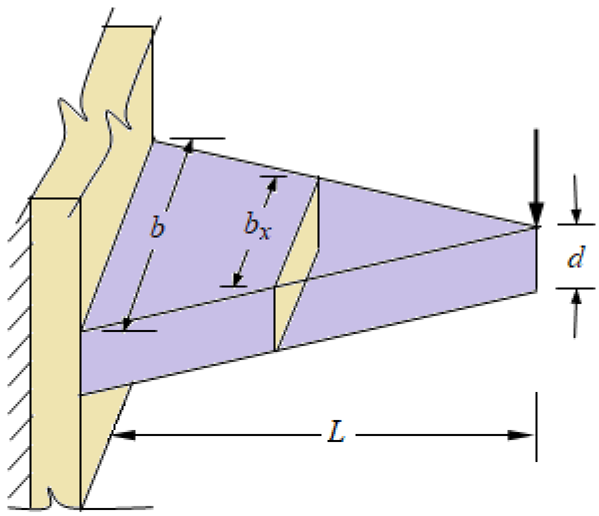


Fig. 20

From expression of flexure, we xhave

$$M = \sigma Z$$

$$Wx = \sigma Z$$

Where σ is the maximum stress at every section of the beam.

If b_x width at any section XX, then $Z = \dfrac{b_x d^2}{6}$

$$\therefore \qquad \sigma = \frac{Wx}{\frac{b_x d^2}{6}} = \frac{6Wx}{b_x d^2}$$

Similarly, maximum stress at support, $\sigma = \dfrac{6WL}{bd^2}$

Equating equation () and (), we have

$$\frac{6Wx}{b_x d^2} = \frac{6WL}{bd^2}$$

$$b_x = b\left(\frac{x}{L}\right)$$

At free end, i.e., $x = 0$, the width of beam $b_0 = 0$

At the fixed end, i.e., $x = L$, the width $b_L = b\left(\frac{L}{L}\right) = b$

By varying the depth of beam and keeping the width constant

Consider a cantilever beam of length *L* and uniform width *b* carrying a concentrated load *W* at its free end as shown in the Fig. 20. Let the depth varies from a minimum at its free end to a maximum of *d* near the fixed end.

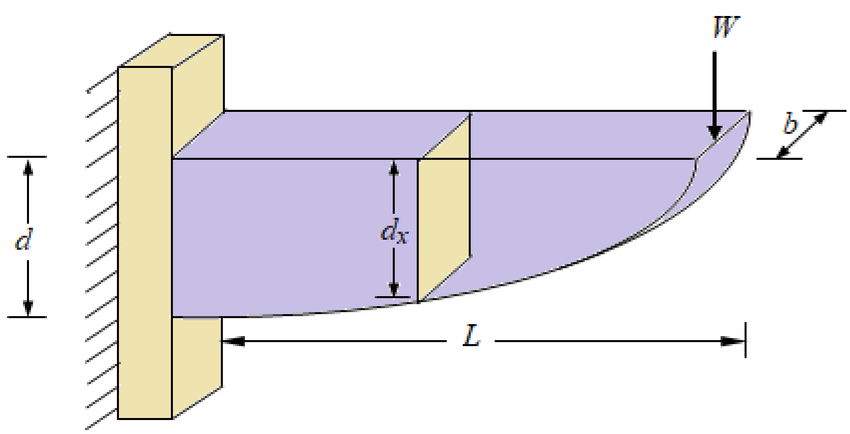


Fig. 21

It is obvious that the bending moment varies from minimum zero at the free end to maximum at WL at the fixed support.

Bending moment at any section at a distance of x from the free end,

$$M = Wx$$

From expression of flexure, we xhave

$$M = \sigma Z$$

$$Wx = \sigma Z$$

Where σ is the maximum stress at every section of the beam.

If b_x width at any section XX, then $Z = \dfrac{bd_x^2}{6}$

$$\therefore \qquad \sigma = \frac{Wx}{\dfrac{bd_x^2}{6}} = \frac{6Wx}{bd_x^2}$$

Similarly, maximum stress at support, $\sigma = \dfrac{6WL}{bd^2}$

Equating equation () and (), we have

$$\frac{6Wx}{bd_x^2} = \frac{6WL}{bd^2}$$

$$d_x = d\sqrt{\left(\frac{x}{L}\right)}$$

At free end, i.e., $x = 0$, the depth of beam, $d_0 = 0$

At the fixed end, i.e., $x = L$, the depth, $d_L = d\sqrt{\left(\dfrac{L}{L}\right)} = d$

Numerical

1. A rectangular beam of breadth 100 *mm* and depth 200 *mm* is simply supported over a span of 4 *m*. The beam is loaded with an uniformly distributed load of 5 *kN/m* over the entire span. Find the maximum bending stresses.

Solution:

Breadth of the beam, $b = 100\ mm$

Depth of beam, $d = 200\ mm$

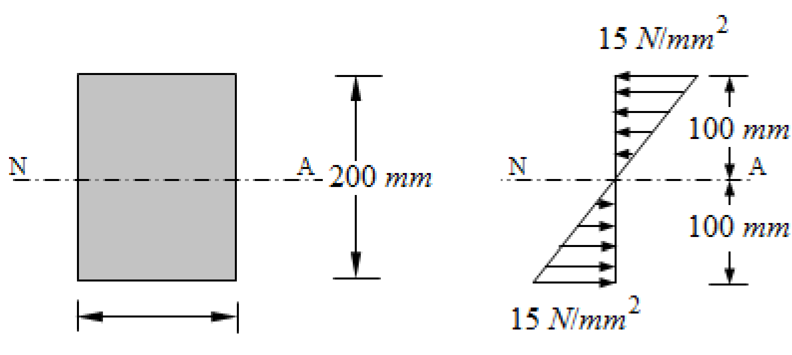
Moment of inertia, $I = \frac{1}{12}bd^3 = \frac{1}{12}\times 100\times(200)^3 = 66.67\times 10^6\ mm^4$

Span of beam, $l = 4\ m$

Uniformly distributed load, $w = 5\ kN/m$

Maximum bending moment at centre of beam, $M = \frac{wl^2}{8} = \frac{5\times 4^2}{8}$

$$= 10\,kN.M = 10^7\,N.mm$$



(a) Cross section of beam (b) Bending stress distribution

Fig. 22

Neutral axis passes through the centroid of section.

The distance of top and bottom fibre from the neutral axis, $y = 100\ mm$

Thus, maximum bending stress, $\sigma = \frac{M}{I}y = \frac{10^7}{66.67\times 10^6}\times 100$

$$= 15\,N/mm^2$$

2. A beam of I-section shown in Fig. 23 is simply supported over a span of 10 *m*. It carries a uniform load of 4 *kN*/*m* over the entire span. Evaluate the maximum bending stresses.

Solution:

Moment of inertia, $I = \frac{1}{12}\left(BD^3 - bd^3\right) = \frac{1}{12}\left(300 \times 660^3 - 280 \times 600^3\right)$

$$= 21.474 \times 10^8\ mm^4$$

Span of the beam, $l = 10\ m$

Uniformly distributed load, $w = 4\ kN/m$

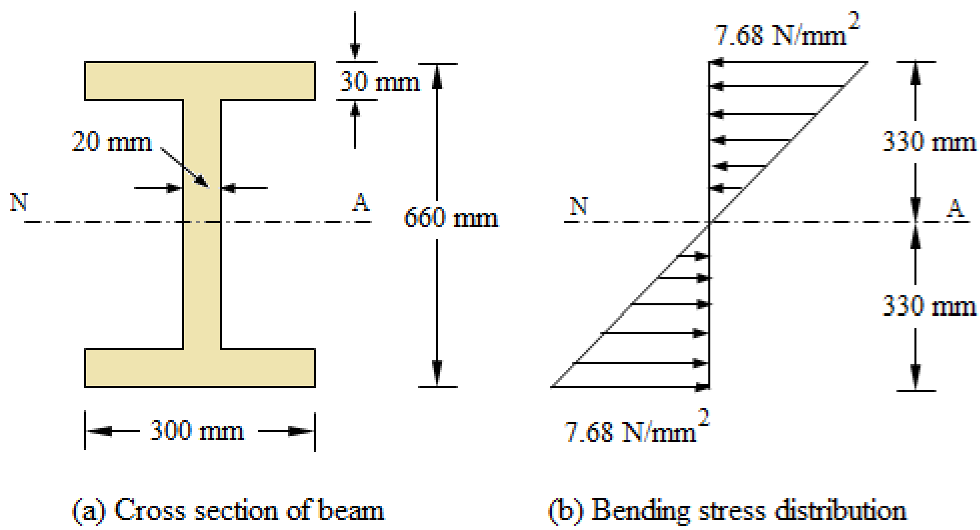


Fig. 23

Maximum bending moment at centre of beam, $M = \frac{4 \times 10^2}{8} = 50\,kN.m$

$$= 5 \times 10^7\ N.mm$$

Neutral axis passes through the centroid of I-section.

The distance of top and bottom fibre from the neutral axis, $y = 330\ mm$

Thus, maximum bending stress, $\sigma = \frac{M}{I} y = \frac{5 \times 10^7}{21.474 \times 10^8} \times 330 = 7.68\, N/mm^2$

The bending stress at top and bottom fibres $= 7.68 \times 10^8\ N/mm^2$

3. A beam of an I-section shown in Fig. 24 is simply supported over a span of 4 *m*. Find the uniformly distributed load the beam can carry if the bending stress is not to exceed 100 N/mm^2.

Solution:

Moment of inertia, $I = \frac{1}{12}\left(BD^3 - bd^3\right) = \frac{1}{12}\left(200 \times 300^3 - 180 \times 260^3\right)$

$$= 180.36 \times 10^6\ mm^4$$

Maximum bending stress, σ_{max} = 100 *N/mrn*^{2}

Span of beam, l = 4 *m*

Extreme fibre distance, y_{max} = 150 *mm*

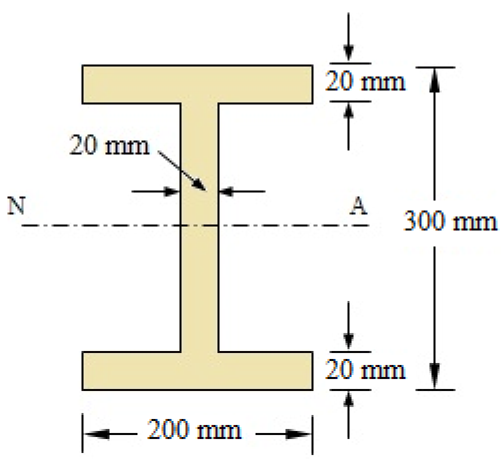


Fig. 24

Section modulus, $Z = \frac{I}{y_{max}} = \frac{180.36 \times 10^6}{150} = 1242400\, mm^3$

Maximum bending moment, $M = \sigma_{max} Z = 100 \times 1242400$

$$= 124240000\, N.mm$$

$$= 124.24\, kN.m$$

But $$M = \frac{wl^2}{8}$$

$$124.24 = \frac{w \times (4)^2}{8}$$

$$w = \frac{124.24 \times 8}{16} = 64.12\, kN/m$$

The maximum uniformly distributed load the beam can carry = 64.12 *kN/m*.

4. A timber beam of rectangular section carries a load of 2 *kN* at mid-span. The beam is simply supported over a span of 3.6 *m*. If the depth of section is to be twice the breadth, and the bending stress is not to exceed 9 N/mrn^2, determine the cross-sectional dimensions.

Solution:

Span of the beam, l = 3.6 *m*

Uniformly distributed load, w = 2 *kN*

Allowable bending stress, σ_{allow} = 9 N/mm^2

Maximum bending moment at centre of beam, $M = \frac{WL}{4} = \frac{2 \times 3.6}{4} = 1.8\, kN.m$

$$= 1.8 \times 10^6\ N.mm$$

From the flexural relationship, we have $$Z = \frac{M}{\sigma_{allow}}$$

$$\frac{1}{6} bd^2 = \frac{1.8 \times 10^6}{9}$$

$$bd^2 = \frac{1.8 \times 10^6}{9} \times 6 = 1.2 \times 10^6$$

Depth of section is to be twice the breadth, i.e., $d = 2b$

So, we have $$b(2b)^2 = 1.2 \times 10^6$$

$$b^3 = \frac{1.2\times10^6}{4} = 0.3\times10^6$$

$$b = 64.94\,mm$$

$$d = 2\times64.943 = 129.886\,mm$$

Therefore, width of beam = 65 *mm*, and depth of beam = 130 *mm*

5. A rectangular beam of width 200 *mm* and depth 300 *mm* is simply supported over a span of 5 *m*. Find the safe uniformly distributed load that the beam can carry per metre length if the allowable bending stress in the beam is 100 N/mm^2.

Solution:

Span of beam, l = 5 *m*

Width Breadth of the beam, b = 100 *mm*

Depth of beam, d = 200 *mm*

Allowable bending stress, σ_{allow} = 100 N/mm^2

Section modulus, $Z = \frac{1}{6}bd^2 = \frac{1}{2}\times200\times300^2 = 3\times10^6\ mm^3$

Moment of resistance of the beam, $M = \sigma_{allow}Z = 100\times3\times10^6$

$$= 300\times10^6\ N.mm = 300\,kN.m$$

Maximum bending moment at the centre of the beam,

$$M = \frac{wl^2}{8}$$

$$300 = \frac{w\times(5)^2}{8}$$

$$\therefore \qquad w = \frac{300\times8}{25} = 96\,kN.m$$

So, the load that the beam can carry is 96 kN/m.

6. A rectangular beam of size 60 *mm* x 100 *mm* has a central rectangular hole of size 15 *mm* x 20 *mm*. The beam is subjected to bending and the maximum bending stress is limited to 100 N/mm^2. Find the moment of resistance of the hollow beam section.

Solution:

External dimension of hollow rectangular beam: B = 60 mm, D = 100 mm

Size of the central hole: b = 15 mm, d = 20 mm

Moment of inertia of the hollow beam section, $I = \frac{1}{12}\left(BD^3 - bd^3\right) = \frac{1}{12}\left(60\times 100^3 - 15\times 20^3\right)$

$$= 4.999\times 10^6\ mm^4$$

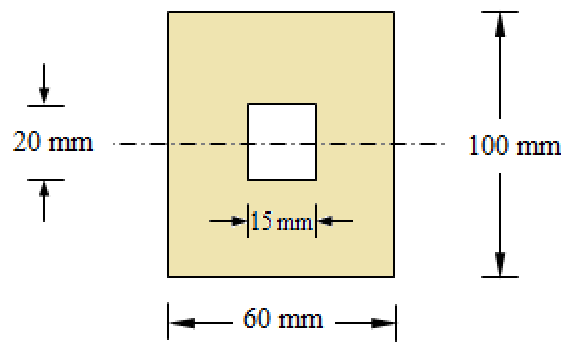


Fig.25

Extreme fibre distance, $y_{max} = \frac{100}{2} = 50\,mm$

Section modulus, $Z = \frac{I}{y_{max}} = \frac{4.999\times 10^6}{50} = 9.98\times 10^4\ mm^3$

Allowable bending stress, σ_{allow} = 100 N/mm^2

Moment of resistance, $M = \sigma_{allow} Z = 100\times 9.98\times 10^4$

$$= 9.98\times 10^6\ N.mm$$

$$= 9.98\,kN.mm$$

7. Find the ratio of the dimensions of the strongest rectangular beam that can be cut from a circular log of wood of diameter *D*.

Solution:

Let b be the width and d the depth of the strongest rectangular beam section as shown in the Fig. 26.

From the geometry, we have $b^2 + d^2 = D^2$

$$d^2 = D^2 - b^2$$

Section modulus of the rectangular section,

$$Z = \frac{1}{6}bd^2 = \frac{1}{6}b\left(D^2 - b^2\right)$$

$$= \frac{1}{6}\left(bD^2 - b^3\right)$$

Strongest section in bending should have largest section modulus.

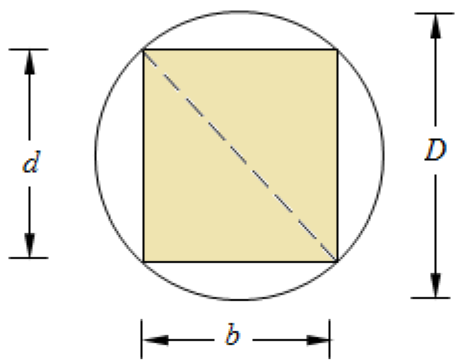


Fig. 26

Hence, $$\frac{dZ}{db} = \frac{1}{6}\left(D^2 - 3b^2\right) = 0$$

$$3b^2 = D^2$$

$$b = \frac{D}{\sqrt{3}}$$

And $$d = \sqrt{D^2 - b^2} = \sqrt{D^2 - \frac{D^2}{3}}$$

$$= \sqrt{\frac{2D^2}{3}} = \left(\sqrt{\frac{2}{3}}\right)D$$

8. Two sections of same material; one of solid circular section and the other hollow circular section of internal diameter half the external diameter, have the same flexural strength. Which one of them is economical?

Solution:

Let D = Diameter of solid circular section

D_1 = Outer diameter of hollow circular section

Inside diameter of hollow circular section, $D_2 = 0.5\,D_1$

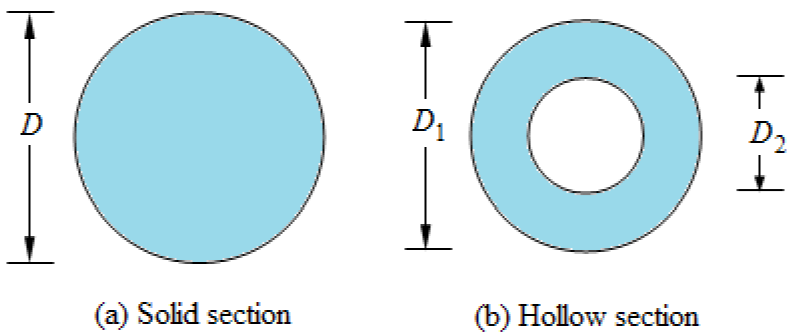


Fig. 27

Section modulus of solid section, $Z_1 = \frac{\pi}{32}D^3$

Section modulus of hollow section, $Z_2 = \frac{\pi}{32D_1}\left(D_1^4 - D_2^4\right) = \frac{\pi}{32D_1}\left\{D_1^4 - \left(0.5D_1\right)^4\right\}$

$$= \frac{\pi}{32} \times 0.9375 D_1^3$$

Since both sections have same flexural strength, their section modulus should be equal.

Hence, $$\frac{\pi}{32}D^3 = \frac{\pi}{32} \times 0.9375 D_1^3$$

$$D^3 = 0.9375 D_1^3$$

$$D = 0.98 D_1$$

$$\frac{\text{Cross-sectional area of solid section}}{\text{Cross-sectional area of hollow section}} = \frac{A_s}{A_h} = \frac{\frac{\pi}{4}D^2}{\frac{\pi}{4}\left(D_1^2 - D_2^2\right)} = \frac{D^2}{\left\{D_1^2 - \left(0.5D_1\right)^2\right\}}$$

$$= \frac{D^2}{0.75 D_1^2} = \frac{1}{0.75} \times \left(\frac{D}{D_1}\right)^2$$

$$= \frac{1}{0.75} \times \left(0.98\right)^2 = 1.28$$

Since the sectional area of hollow section is less than that of solid section, for a given length of the beam, the weight of hollow section will be less. Hence hollow section is economical.

9. A cantilever of 2 *m* length and square section 200 *mm* x 200 *mm*, just fails in bending when a point load of 12 *kN* is placed at its free end. A beam of rectangular cross section of same material, 150 *mm* wide and 300 *mm* deep, is simply supported over a span of 3 *m*. Calculate the maximum concentrated load that the beam can carry at its centre without failure.

Solution:

The two beams with loading conditions are shown in the Fig.

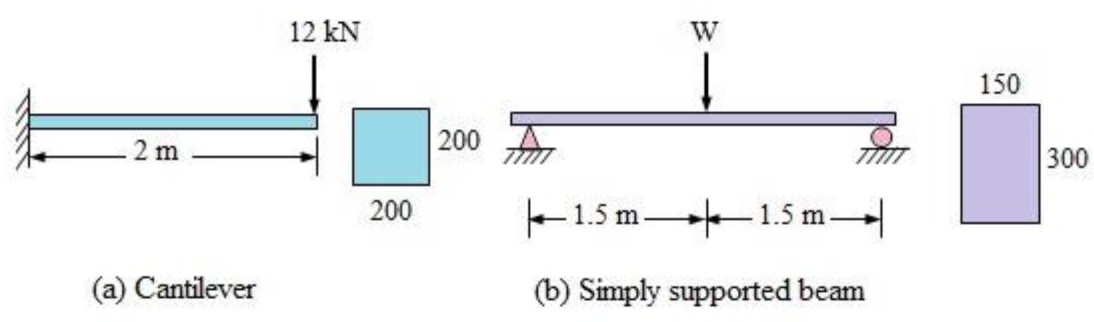


Fig. 28

Maximum bending moment in cantilever beam, $M_c = 12 \times 2 = 24 kN.m$

$$= 24 \times 10^6 \ N.mm$$

Let σ_{allow} is the stress at which the beam fails, $M_c = \sigma_{allow} Z = \frac{1}{6} bd^2 \sigma_{allow}$

$$\frac{1}{6} \times 200 \times 200^2 \times \sigma_{allow} = 24 \times 10^6$$

$$\sigma_{allow} = 18\,N/mm^2$$

Let $W\,kN$ be the maximum central concentrated that the beam can carry without failure.

Maximum bending moment at the mid span, $M_s = \frac{WL}{4} = \frac{W \times 3}{4} = 0.75W\,kN.m$

$$= 0.75 \times 10^6\,W\,N.mm$$

Moment of resistance of simply supported beam section,

$$M_R = \sigma_{allow} Z = 18 \times \frac{1}{6} \times 150 \times 300^2$$

$$= 40.5 \times 10^6\ N.mm$$

Equating maximum bending moment (M_s) to moment of resistance (M_R), we have

$$0.75 \times 10^6\,W = 40.5 \times 10^6$$

$$W = 54\,kN$$

10. For a given sectional area, compare the moments of resistance of circular and square section.

Solution:

Let the diameter of the circular section be d.

Area of circular section, $A = \frac{\pi}{4} d^2$

Section modulus, $Z_C = \frac{\pi}{32} d3$

Let the square section has side of a.

Since both circular and square section have the same area,

$$a^2 = \frac{\pi}{4} d^2$$

$$a = \frac{\sqrt{\pi}}{2} d$$

Section modulus of square section, $Z_S = \frac{a^3}{6} = \frac{\pi\sqrt{\pi}}{48}d^3$

Ratio of Section modulus of square section and circular section,

$$\frac{Z_S}{Z_C} = \frac{\frac{\pi\sqrt{\pi}}{48}d^3}{\frac{\pi}{32}d^3} = 1.18$$

Hence, flexural strength of square section is 1.18 times more than that of circular section of equal area.

11. Compare the moments of resistance of a square section of given material when the beam section is placed such that (i) two sides are parallel and (ii) one diagonal vertical.

Solution:

Square section with two sides horizontal is shown in the Fig. 29(a).

Section modulus of square section with two sides horizontal, $Z_1 = \frac{a^3}{6}$

Let σ is the permissible flexural stress.

Moment of resistance, $M_1 = \sigma Z_1 = \frac{\sigma a^3}{6}$

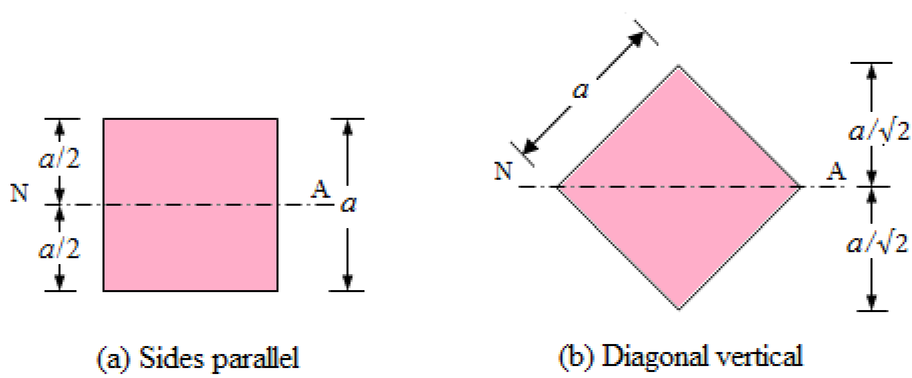


Fig. 29

Square section with on diagonal vertical is shown in the Fig. 29(b).

Moment of inertia about the neutral axis, i.e., the diagonal of the square section = Twice the moment of inertia of triangle of base $\sqrt{2}a$ and height $a/\sqrt{2}$.

$$I_2 = 2 \times \frac{\sqrt{2}a\left(\frac{a}{\sqrt{2}}\right)^3}{12} = \frac{a^4}{12}$$

Extreme fibre distance, $y_{max} = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}$

Section modulus of square section with one diagonal vertical,

$$Z_2 = \frac{I_2}{y_{max}} = \frac{\frac{a^4}{12}}{\frac{a}{\sqrt{2}}} = \frac{\sqrt{2}a^3}{12}$$

Moment of resistance, $M_2 = \sigma Z_2 = \frac{\sqrt{2}\,\sigma\, a^3}{12}$

Ration of the moments of resistance of section in two different positions,

$$\frac{M_1}{M_2} = \frac{\frac{\sigma a^3}{6}}{\frac{\sqrt{2}\sigma a^3}{12}} = \sqrt{2} = 1.414$$

12. Three beams of same material with circular, square and rectangular cross sections have the same length and are subjected to same maximum bending moment. The depth of the rectangular section is twice the width. Compare their weights.

Solution:

Fig. 30 shows three different sections, circular, square, and rectangular of beam.

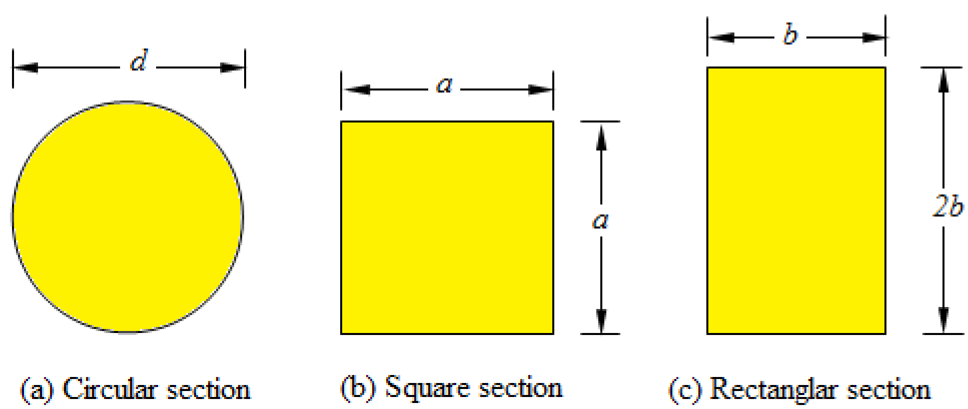


Fig. 30

Let Diameter of circular section = *d*,

Side of square section = *a*, and

Width and depth of rectangular section are *b* and 2*b* respectively.

As beams of three different cross sections of equal allowable stress are subjected same maximum bending moment, they must have same strength. Hence, all sections should have equal section modulii.

Section modulus of circular section, $Z_C = \dfrac{\pi d^3}{32}$

Section modulus of circular section, $Z_S = \dfrac{a^3}{6}$

Section modulus of circular section, $Z_R = \dfrac{b(2b)^2}{6} = \dfrac{2}{3}b^3$

We have $$\frac{\pi d^3}{32} = \frac{a^3}{6} = \frac{2}{3}b^3$$

$$\therefore \quad d = 1.193a \text{ and } b = 0.6299a$$

$$\frac{\text{Weight of circular beam}}{\text{Weight of square beam}} = \frac{\text{Area of circular section}}{\text{Area of square section}} = \frac{\frac{\pi d^2}{4}}{a^2} = \frac{\pi}{4}\left(\frac{d}{a}\right)^2$$

$$= \frac{\pi}{4}(1.193)^3 = 1.118$$

$$\frac{\text{Weight of rectangula r beam}}{\text{Weight of square beam}} = \frac{\text{Area of rectangula r section}}{\text{Area of square section}} = \frac{2b^2}{a^2} = 2\left(\frac{b}{a}\right)^2$$

$$= 2(0.6299)^2 = 0.7936$$

13. A beam of symmetric *I*-section has flange size 100 mm x 15 mm, overall depth 250 mm. Thickness of web is 8 mm. Compare the flexural strength of this section with that of a beam of rectangular section of same material and area. The width of rectangular section is two-third of its depth.

Solution:

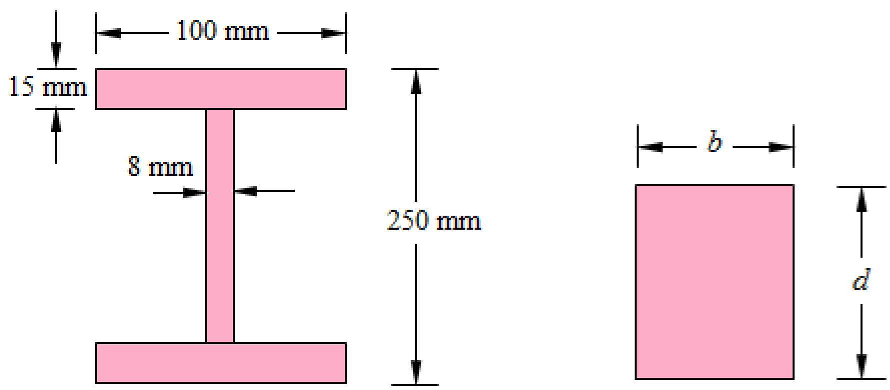
The *I*-section and the rectangular section of equal area are shown in the Fig. 31.

Area of *I*-section, $A_I = (2 \times 100 \times 15) + (220 \times 8) = 4760\,mm^2$

Moment of inertia of *I*-section, $I_I = \dfrac{100 \times 250^3}{12} - \dfrac{92 \times 220^3}{12} = 4.8574\,mm^4$

Section modulus of *I*-section, $Z_I = \dfrac{I}{y_{\max}} = \dfrac{4.8574 \times 10^7}{125}$

$$= 388592\,mm^3$$



(a) I-section (b) Rctangular section

Fig. 31

Let the depth of the rectangular section = *d mm*

Width of the rectangular section, $b = \dfrac{2}{3}d$

Area of the rectangular section, $A_R = \dfrac{2}{3}d \times d = \dfrac{2}{3}d^2$

Since the area of two sections are equal, $\dfrac{2}{3}d^2 = 4760$

$$d = 84.50\,mm$$

and $b = \dfrac{2}{3} \times 84.50 = 56.33\,mm$

Section modulus of rectangular section, $Z_R = \dfrac{bd^2}{6} = \dfrac{56.33 \times (84.50)^2}{6}$

$$= 67035\,mm^3$$

$$\frac{\text{Flexural strength of I-section}}{\text{Flexural strength of rectangular section}} = \frac{Z_I}{Z_R} = \frac{388592}{67035} = 5.80$$

14. A cast iron beam of an *I*-section with top flange 80 *mm* x 40 *mm*, bottom flange 160 *mm* x 40 *mm* and web 120 *mm* x 20 *mm*. If the tensile stress is not to exceed 30 N/mm^2 and compressive stress 90 N/mm^2, what is the maximum uniformly distributed load the beam can carry over a simply supported span of 6 *m*, if the bottom flange is in tension?

Solution:

The cross section of the beam is as shown in the Fig. 32.

Let $\bar{y}$ is the distance of the centroid (neutral axis) from the bottom fibre (tension fibre).

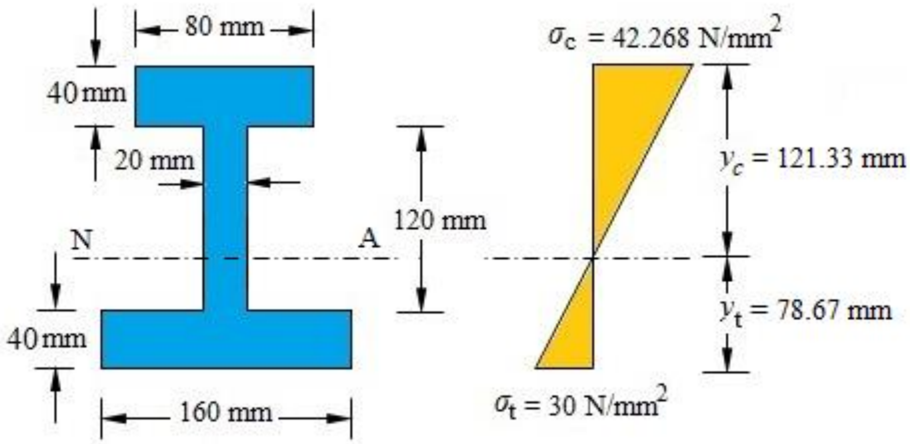


Fig. 32

$$\bar{y} = \frac{\sum a_i y_i}{A} = \frac{160\times 40\times 20 + 20\times 120\times 100 + 80\times 40\times 180}{80\times 40 + 20\times 120 + 80\times 40\times 120}$$

$$= \frac{944000}{12000} = 78.67\,mm$$

Moment of inertia,

$$I = \frac{1}{12}\times 160\times 40^3 + 160\times 40\times (78.67-20)^2 + \frac{1}{12}\times 20\times 120^3 + 20\times 120\times (100-78.67)^2$$
$$+\frac{1}{12}\times 80\times 40^3 + 80\times 40\times (180-78.67)^2$$

$$= 60138670\,mm^4$$

Tension occurs at the bottom and compression at the top.

Bottom extreme fibre distance (large flange, tension flange), $y_t = 78.67\,mm$

Top extreme fibre distance (compression flange), $y_c = 200 - 78.67 = 121.33\,mm$

Moment of resistance from tensile strength consideration,

$$= \sigma_{allow} \frac{I}{y_t} = 30 \times \frac{60138670}{78.67} = 22933266.81\,N.mm$$

$$= 22.933\,kN.m$$

Moment of resistance from compressive strength consideration,

$$= \sigma_{allow} \frac{I}{y_c} = 90 \times \frac{60138670}{121.33} = 44609579.65\,N.mm$$

$$= 44.609\,kN.m$$

Hence, actual moment resistance is smaller of the above two, i.e., 22.993 *kN*

Maximum bending moment, $= \frac{wl^2}{8} = \frac{w \times 6^2}{8} = 4.5w$

Equating the maximum bending moment with the moment of resistance, we have

$$4.5w = 22.933$$

$$w = 5.096\,kN/m$$

Alternatively,

Suppose the maximum stress in compression at the top is 90 N/mm^2.

Corresponding maximum stress in tension at the bottom,

$$\sigma_t = \frac{y_t}{y_c} \times \sigma_c = \frac{78.67}{121.33} \times 90$$

$$= 58.355 > 30\,N/mm^2 \quad \text{(Not possible)}$$

But the permissible tensile stress is only 30 N/mm^2. Hence, let the maximum tensile stress be allowed to reach 30 N/mm^2.

Corresponding maximum compressive stress at the top,

$$\sigma_c = \frac{y_c}{y_t} \times \sigma_t = \frac{121.33}{78.67} \times 30$$

$$= 42.268\, N/mm^2 < 90\, N/mm^2 \ (OK)$$

Hence, the beam will fail in tension at the bottom flange.

Moment of resistance from tensile strength consideration,

$$= \sigma_{allow} \frac{I}{y_t} = 30 \times \frac{60138670}{78.67} = 22933266.81\, N.mm$$

$$= 22.933\, kN.m$$

Maximum bending moment, $= \frac{wl^2}{8} = \frac{w \times 6^2}{8} = 4.5w$

Equating the maximum bending moment with the moment of resistance, we have

$$4.5w = 22.933$$

$$w = 5.096\, kN/m$$

15. Two wooden planks 60 mm x 160 mm each are connected together to form a cross section of a beam as shown in the Fig. If a sagging bending moment of 3500 *N.m* is applied about the horizontal axis, find the stresses at the extreme fibre of the cross-section. Also calculate the total tensile force on the cross-section.

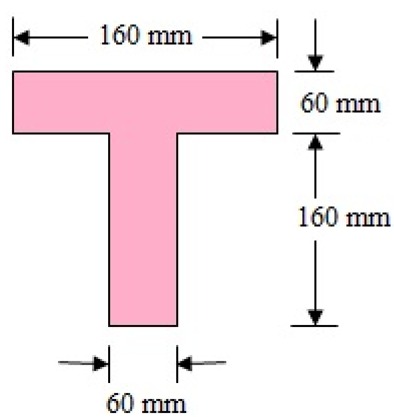


Fig.

Solution:

Let us locate the centroid and hence the neutral axis, and find moment of inertia of the section. Consider the bottom of T-section as the reference axis for location of centroid. The T-section consists of two components, web and flange.

The relevant calculations are shown in the table.

Distance of the centroidal axis GG from the bottom edge,

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{2610000}{19200} = 135.94\,mm$$

Moment of inertia at the bottom edge, $I_b = \sum I_{Self} + \sum ay^2$

$$= 23.36 \times 10^6 + 408 \times 10^6 = 431.36 \times 10^6\ mm^4$$

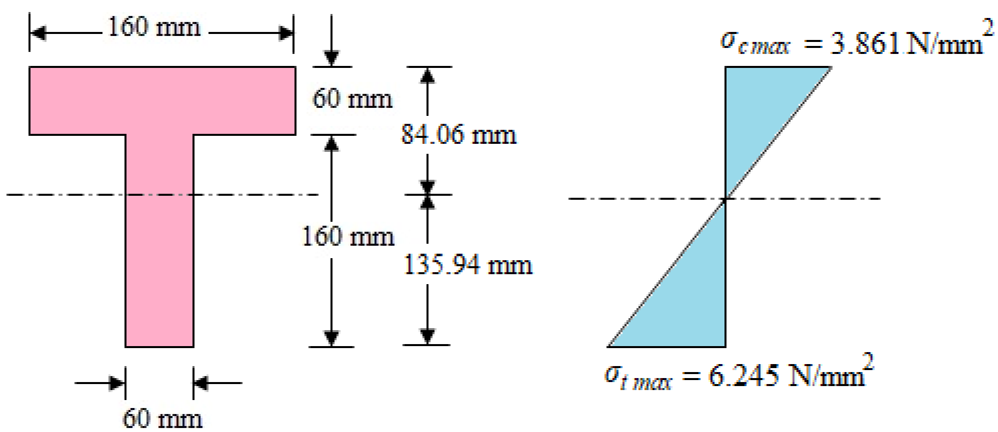


Fig.

But, $$I_b = I_G + \left(\sum a\right)\bar{y}^2$$

$$I_G = I_b - \left(\sum a\right)\bar{y}^2 = 431.36 \times 10^6 - 19200 \times 135.94^2$$

$$= 76190074.88\,mm^4$$

Let the maximum tensile and compressive stresses at extreme fibres be σ_{tmax} and σ_{cmax} respectively.

Components	Area a (mm^2)	Centroidal distance from the bottom edge, y (mm)	ay (mm^3)	ay^2 (mm^4)	I_{Self} (mm^4)
Web	9600	80	786000	61.44 x 10^6	$\frac{60\times160^3}{12}=20.48\times10^6$
Flange	9600	190	1824000	346.56 x 10^6	$\frac{160\times60^3}{12}=2.88\times10^6$
Total	19200		2610000	408 x 10^6	23.36 x 10^6

We have, $$\sigma_{t\max}=\frac{M}{I}y_t=\frac{3500\times1000}{76190074.88}\times135.94$$

$$=6.245\,N/mm^2$$

$$\sigma_{c\max}=\frac{M}{I}y_c=\frac{3500\times1000}{76190074.88}\times84.06$$

$$=3.861\,N/mm^2$$

Total tensile force = Average tensile stress x area of tensile zone

$$=\frac{6.245}{2}\times(135.94\times60)=25468.359\,N$$

16. A water main of 1000 mm internal diameter and 10 mm thickness is running full. If the bending stress is not to exceed 56 N/mm^2, find the greatest span on which the pipe may be freely supported. Steel and water weigh 76800 N/m^3 and 10000 N/m^3 respectively.

Solution:

Internal diameter of the pipe, d = 1000 mm = 1 m

External diameter of pipe, D = 1000 + 2 x 10 = 1020 mm = 1.02 m

Consider 1 m length of the water main.

Area of the pipe section, $A=\frac{\pi}{4}\left(D^2-d^2\right)=\frac{\pi}{4}\left(1.02^2-1^2\right)$

$$=0.03173\,m^2$$

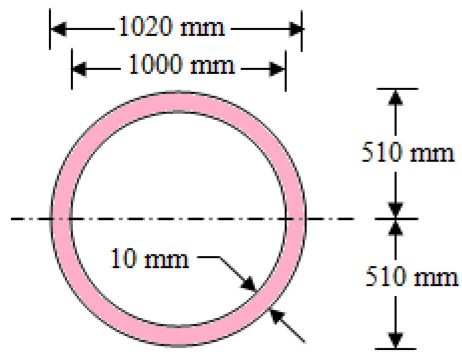


Fig.

Area of the water section, $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 1^2$

$$= 0.7854\, mm^2$$

Weight of one metre length of pipe = 0.03173 x 1 x 76800 = 2493.978 N

Weight of water in one metre length of the pipe = 0.7854 x 1 x 10000 = 7854 N

Total load on the pipe per metre run = 2493.978 + 7854 = 10347.978 N

Let the maximum span of the pipe $l\ m$.

Maximum bending moment, $M = \frac{wl^2}{8} = \frac{10347.978 l^2}{8} = 1293.497 l^2\ N.m$

$$= 1293.497 \times 1000 l^2\, N.mm$$

Moment of inertia of the pipe section about the neutral axis,

$$I = \frac{\pi}{64}\left(D^4 - d^4\right) = \frac{\pi}{64}\left(1020^4 - 1000^4\right)$$

$$= 4046.379 \times 10^6\ mm^4$$

We know, $$\frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{1293.497 \times 1000 l^2}{4046.379 \times 10^6} = \frac{56}{510}$$

$$l^2 = \frac{56 \times 4046.379 \times 10^6}{510 \times 1293.497 \times 1000} = 343.494$$

$$l = 18.533\,m$$